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DERIVATONS OF FORMULAS FOR MEASURES OF EFFECTIVENESS, SAFETY STOCK, AND MIN-COST ORDER AND REPAIR QUANTITIES FOR A READINESS-BASED REPAIRABLE ITEM INVENTORY MODEL FOR THE U.S. NAVY

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A new wholesale level replenishment model is being developed for managing the Navy's inventories of repairable items. It is a readiness-based model that seeks to determine the depths of items of a weapon system which minimize the system's Mean Supply Response Time subject to budget constraint. The model incorporates both a batch procurement and batch repair of the items. Required inputs to this model are the specified values of each. Basic to the development of this model are the derivations of formulas for the probability of being out of stock at any instant of time and the expected number of backorders at any instant of time. Formulas for these measures are presented for the assumptions of both Poisson and Normal demand during the aggregate lead time. The model also needs a formula for the safety stock. Therefore, approximate formulas for safety stock have been derived through the use of a simulation model of the repairable inventory management process. Finally, because the batch procurement and repair quantities are required inputs to the proposed model, formulas for approximate least cost values have been derived as part of a study of possible candidate values for these inputs.

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CHAPTER 1 - INTRODUCTION

A. Background

In the 1960's the Navy installed the first mainframe computers to manage their vast inventories of spare and repair parts. Along with these computers they installed inventory management models which had been developed by Hadley and Whitin [3]. The objective function of these models was the minimization of the average annual total variable costs to procure and hold inventories.

The Navy manages both consumable and repairable items. Consumable items are discarded when they cease to function correctly. For repairable items an attempt is made to repair a nonfunctioning item. The inventory models developed by Hadley and Whitin {3} were for the consumable items. Since there was no model in Reference [3] for repairables the Navy decided to approach the repairable problem by subdividing the problem into two distinct parts, those nonfunctioning units which couldn't be repaired and those that could be repaired. Each part was "managed" using the same model structure as was being used for the consumable items. The units, which could not be repaired, were replaced in batches through a procurement action. Those that could be repaired were batch inducted for repair.

Using the two-part approach, the Navy was able to develop formulas for the economic order and repair quantities. To determine the two reorder points the Navy needed to have a backorder cost. However, they had no way of determining such a cost. Therefore they adopted an approach which was to set a goal of satisfying an average annual requisition fill rate. This measure was called

the "supply material availability" or SMA and the overall goal was an SMA of 85%. From the SMA goal an implied backorder cost could be determined.

In 1982 the Navy attempted to integrate the two parts. Unfortunately, the effort was only partially successful; two inventory management systems still existed.

In the late 1970's the decision was made to upgrade the mainframe computers. The Navy decided that it would also be a good time to review its models and improve them where possible. The Naval Postgraduate School was asked to participate in this model improvement process. In 1984 the Navy accepted a wholesale provisioning model by Richards and McMasters [7] of the Naval Postgraduate School which had a readiness-based objective function. It was the minimization of the aggregate Mean Supply Response Time (MSRT) for a group of new items for a specific weapon system. The model sought maximum inventory levels for the items which would satisfy this objective subject to a provisioning budget constraint. Unfortunately, Navy did not have a replenishment model which was readiness-based. Therefore, the provisioning model was never used. In an attempt to resolve this problem this author began the search for a replenishment model in 1986 which has the same objective function as the provisioning model and a similar budget constraint.

An appropriate model for managing a group of consumable items was developed in 1989 [2]. A model for managing repairable items was more difficult because of the complexity of the process. In 1988 a study of preliminary simulation results suggested to this author that when demand is modeled as a Poisson process that the probability distribution for the inventory position (on-hand + on-order + in repair - backorders) could be approximated by the

convolution of two discrete uniform distributions, one for repairable carcasses and the other for carcasses which were either not returned or not repairable (attritions). Batch procurement of a quantity Q_P and batch induction of a repair quantity Q_R were assumed (see References [1] and [4] for the details of the probability distribution).

The simulation model required further refining based on discussions with operations analysts and repairables managers working for the Navy's Inventory Control Points. The current form of the simulation model for repairables was finalized in 1992. When a Poisson demand occurs the model decides if there is a carcass being returned; if so, then that carcass enters a repair queue; if not, then that information is sent to an attrition queue. When Q_R carcasses have accumulated in the repair queue the entire batch sent to the depot for repair. However, they are usually inducted one at a time. As each is inducted, it is examined to see whether it is capable of being repaired; if not, then an attrition is added to the attrition queue. Carcasses which can be successfully repaired ("good" carcasses) pass through the repair process. The first "good" carcass goes immediately into repair and departs a repair turnaround time (RTAT) later. The second carcass waits a short period of time (as if waiting for the first item to finish the first stage of repair) and then enters the repair process if it can be repaired; otherwise, it is rejected and recorded as an attrition and the next carcass is immediately examined. A "good" carcass completes repair a RTAT later. As each "good" carcass in repair is completed it is returned to the ready-for-issue (RFI) inventory. When the attrition queue reaches a size Q_P a procurement of Q_P units is made and that batch is delivered to the RFI inventory a procurement

lead time (PCLT) later. A flow chart of this model is presented as Figure 1 in Chapter 6 of this report.

Simulation results from this model [4] showed that the approximate inventory position distribution developed by this author was quite robust; that is, it accurately represented the simulated distribution of inventory position for any set of Q_P, Q_R values regardless of changes in values of all the other model parameters.

In 1994, after examining the simulation results of the repairable item inventory management process just described, Baker [1] was able to apply stochastic modeling techniques to derive an approximate formula for the probability distribution for the value of the net inventory at any instant of time. This allowed the next step in the model's development, which was to derive the formulas for the determining of the probability of being out of stock at any instant of time and the expected number of backorders at any instant of time for a repairable item. The importance of knowing the expected number of backorders at any instant of time is that it can be used to determine the MSRT. The probability of being out of stock at any instant of time is needed to determine the expected number of backorders at any instant of time and the SMA.

Two other aspects of the model have also been studied; safety stock and economic order and repair quantities. The Navy uses the safety level (safety stock quantity) to provide a measure of protection provided by an inventory of an item. For a repairable item there exists no theoretical formula, although the Navy does have a procedure for determining one since they need it for

developing reorder points for the two parts of their model. A simulation study was conducted this past year to determine an approximate formula for safety stock which would apply for the process modeled. The definition of this safety stock [4] is that it is "the expected net inventory at the time an order of new units arrives into the RFI inventory and/or a repaired carcass is returned to the RFI inventory." Four approximate formulas were found which fit the simulated results quite well; two for the case of batch repair (no delay between "good" carcasses being inducted), and two for the case of "good" carcasses being inducted one at a time with a delay between inductions. The results of that study are presented in this report.

The formulas for the Navy's economic order and repair quantities are modifications of the classic square-root formula known as the "economic order quantity" which is designed for use with consumable items [3]. They do not change when the reorder point is changed although there is an iterative procedure in Reference [3}, Chapter 4 for doing so. The question which came to mind when thinking about the development of a readiness-based model was what effect does a changing maximum inventory position have on the economic values of the order and repair quantities. Do they change or remain fixed? A study was therefore conducted to answer this question. The results of that study are presented in this report.

B. Objectives and Scope

The first objective of this report is to present the derivations of the formulas for determining the probability of being out of stock at any instant of time and the expected number of backorders at any instant of time for a

repairable item for the net inventory probability distribution derived by Baker. In addition, based on discussions with personnel at the Navy's Inventory Control Point (NAVICP), a modification of the assumption of Poisson demand during the aggregate lead time to a Normally distributed demand was requested. Therefore, the second objective is to present the derivations of the formulas for that modification. A third objective is to present the results of a simulation analysis conducted to discover approximate formulas for safety stock. Finally, the fourth objective is to present the results of economic analyses conducted to determine approximate formulas for the economic order and repair quantities.

C. Preview

Chapter 2 presents the derivations of formulas for the probability of being out of stock at any instant of time under the assumptions of Baker's model. Chapter 3 presents the formulas for the expected number of backorders at any instant of time under the assumptions of Baker's model. Chapter 4 presents the derivations of the formulas for the probability of being out of stock at any instant of time under the assumption of a Normal demand during lead time. Chapter 5 presents the derivations for the formulas of the expected number of backorders at any instant of time under the assumption of a Normal demand during lead time. Chapter 6 presents the details of the simulation study used to derive approximate formulas for safety stock for repairable items. Chapter 7 presents the details of the development of approximate formulas for determining the order and repair quantities which minimize the average annual total variable costs of managing an inventory for any specified value of the maximum inventory position. Chapter 8 presents a brief summary of the previous chapters,

conclusions with respect to the chapter results, and recommendations for further model development.

CHAPTER 2 - DERIVATION OF $P_{OUT}(SW)$ FOR BAKER'S MODEL

A. Introduction

This chapter presents the derivations of the formulas for the probability of being out of stock at any instant of time based on the net inventory distribution developed by Baker [1]. To facilitate the process, Baker's distribution is first converted to a form involving complementary distribution functions of the Poisson distribution. This conversion then allows the use of identities from Appendix 3 of Hadley and Whitin [3] in the derivations of the $P_{OUT}(SW)$ formulas.

B. Conversion of the Probability Mass Function of Baker's Model

Equation (38) from Baker's thesis [1] for the case where demand during the aggregate lead time is Poisson was

$$
p[N(t)=SW-z]=\begin{cases}
\dfrac{1}{Q_PQ_R}\displaystyle\sum_{j=0}^{z}\frac{(j+1)\mu^{z-j}e^{-\mu}}{(z-j)!} & for\, 0\le z\le x_1\\[2ex]
\dfrac{1}{Q_PQ_R}\displaystyle\sum_{j=0}^{x_1}\frac{(j+1)\mu^{z-j}e^{-\mu}}{(z-j)!} + \dfrac{min(Q_P,Q_R)}{Q_PQ_R}\displaystyle\sum_{j=x_1+1}^{z}\frac{\mu^{z-j}e^{-\mu}}{(z-j)!} & for\, x_1 < z \le x_2\\[2ex]
\dfrac{1}{Q_PQ_R}\displaystyle\sum_{j=0}^{x_1}\frac{(j+1)\mu^{z-j}e^{-\mu}}{(z-j)!} + \dfrac{min(Q_P,Q_R)}{Q_PQ_R}\displaystyle\sum_{j=x_1+1}^{x_2}\frac{\mu^{z-j}e^{-\mu}}{(z-j)!} + \dfrac{1}{Q_PQ_R}\displaystyle\sum_{j=x_2+1}^{min(x_{Max},z)}\frac{(x_{Max}+1-j)\mu^{z-j}e^{-\mu}}{(z-j)!} & for\, x_2 < z;
\end{cases}
\tag{2.1}
$$

where

$$
x_1 = min(Q_P, Q_R) - 1
$$

$$
x_2 = x_{Max} - x_1 \tag{2.2}
$$

$$
x_{Max} = Q_P + Q_R - 2,
$$

N(t) is the net inventory at any instant of time, and SW is the maximum inventory position. As shown in Baker's thesis, μ = Z where Z is the Program Problem Variable or the modified form ZB which includes REP. The format of equation (2.1) is difficult to use to derive formulas for $P_{OUT}(SW)$ and the time-weighted units short, $B(SW)$.

The following form facilitates the use of identities from Appendix B of Hadley and Whitin [3]. In this form P(.) is the Poisson complementary cumulative distribution function and μ is the mean of the distribution.

$$
Q_P Q_R p(SW - z) = \begin{cases} (z+1-\mu) - (z+1)P(x \geq z+1) + \mu P(x \geq z) & \text{for } 0 \leq z \leq x_1; \\ \\ (x_1+1) - (z+1)P(x \geq z+1) + \mu P(x \geq z) & \\ +(z - x_1)P(x \geq z - x_1) - \mu P(x \geq z - x_1 - 1) & \text{for } x_1 < z \leq x_2; \\ \\ -(z+1)P(x \geq z+1) + \mu P(x \geq z) & \\ +(z - x_1)P(x \geq z - x_1) - \mu P(x \geq z - x_1 - 1) & \\ +(z - x_2)P(x \geq z - x_2) - \mu P(x \geq z - x_2 - 1) & \text{for } x_2 < z. \\ -(z - x_{Max} - 1)P(x \geq z - min(x_{Max}, z)) & \\ +\mu P(x \geq z - min(x_{Max}, z) - 1) & \end{cases}
$$

(2.3)

The derivation of equation (2.3) follows. Beginning with the first term of equation (2.1)

$$
\sum_{j=0}^{z} \frac{(j+1)\mu^{z-j} e^{-\mu}}{(z-j)!},
$$

let x = z - j; then j = z - x and j +1 = z – x +1. For the limits of the summation, when j = 0 then x = z and when j = z then x = 0. As a consequence,

$$
\sum_{j=0}^{z} \frac{(j+1)\mu^{z-j} e^{-\mu}}{(z-j)!} = \sum_{x=0}^{z} (z-x+1) \frac{\mu^x e^{-\mu}}{x!}
$$

$$
= (z+1) \sum_{x=0}^{z} p(x;\mu) - \sum_{x=0}^{z} x p(x;\mu), \qquad (2.4)
$$

where p(x; μ) is the Poisson probability mass function; that is,

$$
p(x;\mu) = \frac{\mu^x e^{-\mu}}{x!}.
$$

Notice that the summation limits when x replaces z - j are reversed. The summation upper limit for j corresponds to the lower limit for x and visa versa.

This equivalence can be seen by expanding the two summations. When this is done we get

$$\sum_{j=0}^{z}\frac{(j+1)\mu^{z-j}e^{-\mu}}{(z-j)!}$$

$$=\frac{\mu^{z}e^{-\mu}}{z!}+\frac{2\mu^{z-1}e^{-\mu}}{(z-1)!}+\frac{3\mu^{z-2}e^{-\mu}}{(z-2)!}+\ldots+\frac{(z-1)\mu^{2}e^{-\mu}}{2!}+\frac{z\mu e^{-\mu}}{1!}+\frac{(z+1)e^{-\mu}}{0!};$$

and

$$\sum_{x=0}^{z}(z-x+1)\frac{\mu^{x}e^{-\mu}}{x!}$$

$$=\frac{(z+1)e^{-\mu}}{0!}+\frac{z\mu e^{-\mu}}{1!}+\frac{(z-1)\mu^{2}e^{-\mu}}{2!}+\ldots+\frac{3\mu^{z-2}e^{-\mu}}{(z-2)!}+\frac{2\mu^{z-1}e^{-\mu}}{(z-1)!}+\frac{\mu^{z}e^{-\mu}}{z!}.$$

The first term of the second line of equation (2.4) can be easily rewritten as

$$\begin{aligned}(z+1)\sum_{x=0}^{z}p(x;\mu)&=(z+1)[P(x\geq 0)-P(x\geq z+1]\\&=(z+1)[1.0-P(x\geq z+1)],\end{aligned}$$

since $P(x \geq 0) = 1.0$ for the Poisson distribution.

The second term of the second line of equation (2.4) can be simplified using Identity 1 from Appendix 3 of Hadley and Whitin [3]

$$\sum_{x=0}^{z}xp(x;\mu)=\sum_{x=0}^{z}\mu p(x-1;\mu)=\mu\sum_{x=0}^{z}p(x-1;\mu)=\mu[1.0-P(x\geq z)].$$

Finally, then,

$$\sum_{j=0}^{z}\frac{(j+1)\mu^{z-j}e^{-\mu}}{(z-j)!}=(z+1-\mu)-(z+1)P(x\geq z+1)+\mu P(x\geq z). \qquad (2.5)$$

For the range of $x_1 < z \leq x_2$ the probability mass function given by equation (2.1) adds a term and changes one of the bounds on the summation of the first term. The first term is now

$$\sum_{j=0}^{x_1} \frac{(j+1)\mu^{z-j} e^{-\mu}}{(z-j)!} = (z+1) \sum_{x=z-x_1}^{z} p(x;\mu) - \sum_{x=z-x_1}^{z} xp(x;\mu)$$

$$= (z+1)[P(x \geq z-x_1) - P(x \geq z+1)] \qquad (2.6)$$
$$- \mu P(x \geq z - x_1 - 1) + \mu P(x \geq z).$$

The second term is

$$\min(Q_P, Q_R) \sum_{j=x_1+1}^{z} \frac{\mu^{z-j} e^{-\mu}}{(z-j)!} = (x_1+1) \sum_{x=0}^{z-x_1-1} \frac{\mu^x e^{-\mu}}{x!} \qquad (2.7)$$
$$= (x_1+1)[1.0 - P(x \geq z - x_1)].$$

Note that $\min(Q_P, Q_R) - 1 \equiv x_1$ (see equation(s)(2.2))

Summing equations (2.6) and (2.7) and collecting like terms yields

$$\sum_{j=0}^{x_1} \frac{(j+1)\mu^{z-j} e^{-\mu}}{(z-j)!} + \min(Q_P, Q_R) \sum_{j=x_1+1}^{z} \frac{\mu^{z-j} e^{-\mu}}{(z-j)!}$$
$$= (x_1+1) - (z+1)P(x \geq z+1) + \mu P(x \geq z) \qquad (2.8)$$
$$+ (z-x_1)P(x \geq z - x_1) - \mu P(x \geq z - x_1 - 1).$$

For $z > x_2$, the first line is the same as for $x_1 < z \leq x_2$. The second line has the upper limit as x_1 instead of z.

$$\min(Q_P, Q_R) \sum_{j=x_1+1}^{x_2} \frac{\mu^{z-j} e^{-\mu}}{(z-j)!} = (x_1+1) \sum_{x=z-x_2}^{z-x_1-1} \frac{\mu^x e^{-\mu}}{x!} \qquad (2.9)$$
$$= (x_1+1)[P(x \geq z - x_2) - P(x \geq z - x_1)].$$

The third line takes a little more work. First,

$$\sum_{j=x_2+1}^{\min(z,x_{Max})} \frac{(x_{Max}+1-j)\mu^{z-j}e^{-\mu}}{(z-j)!} = \sum_{x=z-\min(z,x_{Max})}^{z-x_2-1} \frac{(x_{Max}+1-z+x)\mu^{x}e^{-\mu}}{x!}$$

$$= (x_{Max}+1-z) \sum_{x=z-\min(z,x_{Max})}^{z-x_2-1} p(x;\mu) + \sum_{x=z-\min(z,x_{Max})}^{z-x_2-1} x\,p(x;\mu).$$

Applying Identity 1 of Appendix 3 of Reference [3] leads to

$$\begin{aligned}
&\sum_{j=x_2+1}^{\min(z,x_{Max})} \frac{(x_{Max}+1-j)\mu^{z-j}e^{-\mu}}{(z-j)!} \\
&\quad = (x_{Max}+1-z)[P(x \geq z-\min(z,x_{Max}))-P(x \geq z-x_2)] \\
&\qquad + \mu \sum_{x=z-\min(z,x_{Max})}^{z-x_2-1} p(x-1;\mu) \\
&\quad = (x_{Max}+1-z)[P(x \geq z-\min(z,x_{Max}))-P(x \geq z-x_2)] \\
&\qquad + \mu[P(x \geq z-\min(z,x_{Max})-1)-P(x \geq z-x_2-1)].
\end{aligned} \tag{2.10}$$

Summing equations(2.6), (2.9), and (2.10) and collecting like terms gives the density function for $z > x_2$ in the revised form shown in equation(2.3).

$$\begin{aligned}
&\sum_{j=0}^{x_1} \frac{(j+1)\mu^{z-j}e^{-\mu}}{(z-j)!} + \min(Q_P,Q_R) \sum_{j=x_1+1}^{x_2} \frac{\mu^{z-j}e^{-\mu}}{(z-j)!} + \sum_{j=x_2+1}^{\min(z,x_{Max})} \frac{(x_{Max}+1-j)\mu^{z-j}e^{-\mu}}{(z-j)!} \\
&\quad = -(z+1)P(x \geq z+1) + \mu P(x \geq z) \\
&\qquad + (z-x_1)P(x \geq z-x_1) - \mu P(x \geq z-x_1-1) \\
&\qquad + (z-x_2)P(x \geq z-x_2) - \mu P(x \geq z-x_2-1) \\
&\qquad - (z-x_{Max}-1)P(x \geq z-\min(x_{Max},z)) + \mu P(x \leq z-\min(x_{Max},z)-1).
\end{aligned}$$

C. Derivation of $P_{OUT}(SW)$

1. The case of $0 \leq SW \leq x_1$

The general formula for $P_{OUT}(SW)$ is

$$P_{OUT}(SW) = \sum_{z=SW}^{\infty} [N(t) = SW - z)]. \tag{2.11}$$

Note that when z = SW then N(t) = 0 ; that is, there is no stock on hand. Obviously then, there will be no stock on hand when $z > SW$; the system will be in backorder status with the number of backorders being given by $z - SW$.

The next step is to apply the equation (2.3) results to equation(2.11). We begin with the case for $0 \le SW \le x_1$.

$$P_{OUT}(SW) = \frac{1}{Q_P Q_R} \sum_{z=SW}^{x_1} \left[(z+1-\mu)-(z+1)P(x \ge z+1)+\mu P(x \ge z)\right]$$

$$+\frac{1}{Q_P Q_R} \sum_{z=x_1+1}^{x_2} \begin{bmatrix} (x_1+1)-(z+1)Px \ge z+1)+\mu P(x \ge z) \\ +(z-x_1)P(x \ge z-x_1)-\mu P(x \ge z-x_1-1) \end{bmatrix}$$

$$+\frac{1}{Q_P Q_R} \sum_{z=x_2+1}^{\infty} \begin{bmatrix} -(z+1)P(x \ge z+1)+\mu P(x \ge z) \\ +(z-x_1)P(x \ge z-x_1)-\mu P(x \ge z-x_1-1) \\ +(z-x_2)P(x \ge z-x_2)-\mu P(x \ge z-x_2-1) \\ -(z-x_{Max}-1)P(x \ge z-min\{z,x_{Max}\}) \\ +\mu P(x \ge z-min\{z,x_{Max}\}-1) \end{bmatrix}.$$

(2.12)

This can be further simplified by collecting common terms. For example,

$$-(z+1)P(x \ge z+1)+\mu P(x \ge z)$$

appears in all three parts of equation (2.12). Therefore, for this case it can be summed for z from SW to ∞; that is,

$$\sum_{z=SW}^{\infty} [-(z+1)P(x \ge z+1)+\mu P(x \ge z)].$$

Next, the term

$$(z-x_1)P(x \geq z-x_1) - \mu P(x \geq z-x_1-1)$$

shows up in the second and third parts, so it can be summed for z from x_1 to ∞.

$$\sum_{z=x_1+1}^{\infty} \left[(z-x_1)P(x \geq z-x_1) - \mu P(x \geq z-x_1-1)\right].$$

The last terms come from the third part for which the upper summation bound for z is already ∞. From there we have

$$\sum_{z=x_2+1}^{\infty} \begin{bmatrix} (z-x_2)P(x \geq z-x_2) - \mu P(x \geq z-x_2-1) \\ -(z-x_{Max}-1)P(x \geq z-min\{z,x_{Max}\}) \\ +\mu P(x \geq z-min\{z,x_{Max}\}-1) \end{bmatrix}.$$

There are also the constant terms. From the first part we have

$$\sum_{z=SW}^{x_1}(z+1-\mu) = \sum_{z=SW}^{x_1} z + \sum_{z=SW}^{x_1}(1-\mu)$$
$$= \left[\frac{x_1(x_1+1)}{2} - \frac{(SW-1)SW}{2}\right] + (1-\mu)\left[x_1 - max(0,SW-1)\right].$$

For the second part we have

$$\sum_{z=x_1+1}^{x_2}(x_1+1) = (x_1+1)(x_2-x_1) \quad .$$

Thus, we can rewrite P_{OUT} as

$$P_{OUT}(SW) = \frac{1}{Q_P Q_R}\left[\frac{x_1(x_1+1)}{2} - \frac{(SW-1)SW}{2}\right]$$
$$+\frac{1}{Q_P Q_R}\left[(1-\mu)\left[x_1 - max(0,SW-1)\right] + (x_1+1)(x_2-x_1)\right]$$
$$+\frac{1}{Q_P Q_R}\sum_{z=SW}^{\infty}\left[-(z+1)P(x \geq z+1) + \mu P(x \geq z)\right]$$
$$+\frac{1}{Q_P Q_R}\sum_{z=x_1+1}^{\infty}\left[(z-x_1)P(x \geq z-x_1) - \mu P(x \geq z-x_1-1)\right]$$
$$+\frac{1}{Q_P Q_R}\sum_{z=x_2+1}^{\infty}\begin{bmatrix}(z-x_2)P(x \geq z-x_2) - \mu P(x \geq z-x_2-1) \\ -(z-x_{Max}-1)P(x \geq z-min\{z,x_{Max}\}) \\ +\mu P(x \geq z-min\{z,x_{Max}\}-1)\end{bmatrix}. \tag{2.13}$$

The first two lines of equation (2.13) is consists of the constant terms (i.e., no probabilities need be evaluated). The rest of the equation will require identities from Reference [3] to simplify the form of the summations.

The simplest form is the last term in the third line; namely,

$$\sum_{z=SW}^{\infty} \mu P(x \geq z) = \mu \sum_{z=SW}^{\infty} P(x \geq z) = \mu\alpha(SW). \tag{2.14}$$

For convenience, we have used the following shorthand notation in equation (2.14);

$$\alpha(u) \equiv \sum_{z=u}^{\infty} P(x \geq z),$$

and, from Identity 6 of Appendix 3 of Reference [3], it follows that

$$\alpha(u) \equiv \mu P(x \geq u-1) - (u-1)P(x \geq u). \tag{2.15}$$

As a consequence of the definition of $\alpha(u)$ it follows that the other terms in equation (2.13) involving μ can be readily written as forms of $\alpha(u)$.

$$-\sum_{z=x_1+1}^{\infty} \mu P(x \geq z-x_1-1) = -\mu \sum_{z=x_1+1}^{\infty} P(x \geq z-x_1-1); \qquad (2.16)$$

Let $y=z-x_1-1$. Then, if $z=x_1+1, y=0$. If $z=\infty, y=\infty$. Therefore, we can write equation (2.16) as

$$-\mu \sum_{z=x_1+1}^{\infty} P(x \geq z-x_1-1) = -\mu \sum_{y=0}^{\infty} P(x \geq y) = -\mu\alpha(0).$$

Likewise, we can let $y=z-x_2-1$ in the last part of the fifth line and get

$$-\mu \sum_{z=x_2+1}^{\infty} P(x \geq z-x_2-1) = -\mu \sum_{y=0}^{\infty} P(x \geq y) = -\mu\alpha(0). \qquad (2.17)$$

From the definition of $\alpha(u)$ it follows that

$$\alpha(0) = \mu P(x \geq -1) + P(x \geq 0) = \mu + 1,$$

since $P(x \geq -1) = P(x \geq 0) = 1.0$ for the Poisson distribution.

The next term to be simplified is the first term in the third line of equation (2.13). The process first sets $y=z+1$. Then, when $z=SW$, $y=SW+1$.

Therefore,

$$\sum_{z=SW}^{\infty} (z+1)P(x \geq z+1) = \sum_{y=SW+1}^{\infty} yP(x \geq y) = \gamma(SW+1),$$

where, using Identity 8 of Appendix 3 of Reference [3],

$$\gamma(u) \equiv \sum_{y=u}^{\infty} yP(x \geq y) = \frac{\mu^2}{2}P(x \geq u-2) + \mu P(x \geq u-1) - \frac{u(u-1)}{2}P(x \geq u). \qquad (2.18)$$

When we let $y=z-x_1$ and $y=z-x_2$, respectively, equation (2.18) allows us to write

$$\sum_{z=x_1+1}^{\infty}(z-x_1)P(x\geq z-x_1)=\sum_{y=1}^{\infty}yP(x\geq y)=\gamma(1);$$

$$\sum_{z=x_2+1}^{\infty}(z-x_2)P(x\geq z-x_2)=\sum_{y=1}^{\infty}yP(x\geq y)=\gamma(1). \qquad (2.19)$$

Two terms still remain to be simplified. These are the sixth and seventh lines; namely,

$$\sum_{z=x_2+1}^{\infty}(z-x_{Max}-1)P(x\geq z-\min\{z,x_{Max}\});$$

$$\sum_{z=x_2+1}^{\infty}\mu P(x\geq z-\min\{z,x_{Max}\}-1).$$

It is useful to separate the summation into two parts; from x_2+1 to x_{Max}, and from $x_{Max}+1$ to ∞. The first summations result in

$$(x_{Max}+1)\sum_{z=x_2+1}^{x_{Max}}P(x\geq z-\min\{z,x_{Max}\})=(x_{Max}+1)\sum_{z=x_2+1}^{x_{Max}}P(x\geq 0)=(x_{Max}+1)\sum_{z=x_2+1}^{x_{Max}}1$$
$$=(x_{Max}+1)(x_{Max}-x_2)=(x_{Max}+1)x_1;$$

$$\sum_{z=x_2+1}^{x_{Max}}\mu P(x\geq z-\min\{z,x_{Max}\}-1)=\mu\sum_{z=x_2+1}^{x_{Max}}P(x\geq -1)=\mu(x_{Max}-x_2)=\mu x_1;$$

and

$$-\sum_{z=x_2+1}^{x_{Max}} zP(x \geq z - min\{z, x_{Max}\}) = -\sum_{z=x_2+1}^{x_{Max}} zP(x \geq 0)$$

$$= -\sum_{z=x_2+1}^{x_{Max}} z = -\left[\frac{x_{Max}(x_{Max}+1)}{2} - \frac{x_2(x_2+1)}{2}\right].$$

Putting these parts together as they are in equation (2.13) gives

$$\begin{aligned}
&-\sum_{z=x_2+1}^{x_{Max}} zP(x \geq z - min\{z, x_{Max}\}) \\
&+(x_{Max}+1)\sum_{z=x_2+1}^{x_{Max}} P(x \geq z - min\{z, x_{Max}\}) \\
&+\sum_{z=x_2+1}^{x_{Max}} \mu P(x \geq z - min\{z, x_{Max}\} - 1) \\
&= x_1(x_{Max}+1+\mu) - \left[\frac{x_{Max}(x_{Max}+1)}{2} - \frac{x_2(x_2+1)}{2}\right].
\end{aligned}$$

The final part is to simplify these components for the summations from $x_{Max}+1$ to ∞.

$$\begin{aligned}
&\sum_{z=x_{Max}+1}^{\infty} (-z + x_{Max}+1)P(x \geq z - min\{z, x_{Max}\}) \\
&= -\sum_{z=x_{Max}+1}^{\infty} (z - x_{Max})P(x \geq z - x_{Max}) + \sum_{z=x_{Max}+1}^{\infty} P(x \geq z - x_{Max}) \qquad (2.20) \\
&= -\sum_{y=1}^{\infty} yP(x \geq y) + \sum_{y=1}^{\infty} P(x \geq y) = -\gamma(1) + \alpha(1).
\end{aligned}$$

Here we set $y = z - x_{Max}$. When $z = x_{Max} + 1$ then $y = 1$.

Next,

$$\sum_{z=x_{Max}+1}^{\infty} \mu P(x \geq z - min\{z, x_{Max}\} - 1) = \mu \sum_{z=x_{Max}+1}^{\infty} P(x \geq z - x_{Max} - 1)$$
$$= \mu \sum_{y=0}^{\infty} P(x \geq y) = \mu\,\alpha(0). \tag{2.21}$$

For this part we set $y = z - x_{Max} - 1$ so when $z = x_{Max} + 1$ then $y = 0$.

Finally, combining these two parts gives

$$- \sum_{z=x_{Max}+1}^{\infty} z P(x \geq z - min\{z, x_{Max}\}) + (x_{Max} + 1) \sum_{z=x_{Max}+1}^{\infty} P(x \geq z - min\{z, x_{Max}\})$$
$$+ \sum_{z=x_{Max}+1}^{\infty} \mu P(x \geq z - min\{z, x_{Max}\} - 1) = -\gamma(1) + \alpha(1) + \mu\alpha(0). \tag{2.22}$$

In conclusion, the sum of the last two lines of equation (2.13) is

$$- \sum_{z=x_2+1}^{\infty} (z - x_{Max} - 1) P(x \geq z - min\{z, x_{Max}\}) + \sum_{z=x_2+1}^{\infty} \mu P(x \geq z - min\{z, x_{Max}\} - 1)$$
$$= x_1(x_{Max} + 1 + \mu) - \left[\frac{x_{Max}(x_{Max} + 1)}{2} - \frac{x_2(x_2 + 1)}{2}\right] - \gamma(1) + \alpha(1) + \mu\alpha(0).$$

Earlier we showed a simplified form for $\alpha(0)$. However, for programming purposes it is convenient to leave it in the $\alpha(u)$ general format. That will also be true for $\gamma(1)$. The definitions for $\alpha(u)$ and $\gamma(u)$ were given in equations (2.15) and (2.18), respectively.

$$P_{OUT}(SW)=\frac{1}{Q_P Q_R}\left\{\begin{array}{l}\left[\frac{x_1(x_1+1)}{2}-\frac{(SW-1)SW}{2}\right]\\ +(1-\mu)\left[x_1-max(SW-1)\right]+(x_1+1)(x_2-x_1)\\ -\gamma(SW+1)+\mu\alpha(SW)+\gamma(1)-\mu\alpha(0)\\ +x_1(x_{Max}+1+\mu)-\left[\frac{x_{Max}(x_{Max}+1)}{2}-\frac{x_2(x_2+1)}{2}\right]\\ +\alpha(1)\end{array}\right\}. \quad (2.23)$$

Note that summing the resulting simplifications of equations (2.17), (2.19), and (2.20) gives

$$-\mu\alpha(0)+\gamma(1)-\gamma(1)+\alpha(1)+\mu\alpha(0)=\alpha(1)$$

for the last line of equation (2.23).

2. The case of $x_1 < SW \le x_2$

The next case for $P_{OUT}(SW)$; namely, that for $x_1 < SW \le x_2$, has only the second and third parts of equation (2.13) and the lower bound on the second part is now SW.

$$P_{OUT}(SW)=\frac{1}{Q_P Q_R}\sum_{z=SW}^{x_2}\left[\begin{array}{l}(x_1+1)-(z+1)Px\ge z+1)+\mu P(x\ge z)\\ +(z-x_1)P(x\ge z-x_1)-\mu P(x\ge z-x_1-1)\end{array}\right]$$

$$+\frac{1}{Q_P Q_R}\sum_{z=x_2+1}^{\infty}\left[\begin{array}{l}-(z+1)P(x\ge z+1)+\mu P(x\ge z)\\ +(z-x_1)P(x\ge z-x_1)-\mu P(x\ge z-x_1-1)\\ +(z-x_2)P(x\ge z-x_2)-\mu P(x\ge z-x_2-1)\\ -(z-x_{Max}-1)P(x\ge z-min\{z,x_{Max}\})\\ +\mu P(x\ge z-min\{z,x_{Max}\}-1)\end{array}\right]. \quad (2.24)$$

The constant term in the first line is

$$\sum_{z=SW}^{x_2} (x_1+1) = (x_1+1)(x_2-SW+1). \tag{2.25}$$

Again, for the second and third components of the first line we have

$$\sum_{z=SW}^{\infty} \left[-(z+1)P(x \geq z+1) + \mu P(x \geq z)\right] = -\gamma(SW+1) + \mu\alpha(SW). \tag{2.26}$$

since these parts are also the third line.

The second line of equation (2.24) is now

$$\begin{aligned}
&\sum_{z=SW}^{\infty} \left[(z-x_1)P(x \geq z-x_1) - \mu P(x \geq z-x_1-1)\right] \\
&= \sum_{y=SW-x_1}^{\infty} yP(x \geq y) - \mu \sum_{y=SW-x_1-1}^{\infty} P(x \geq y) \qquad (2.27) \\
&= \gamma(SW-x_1) - \mu\,\alpha(SW-x_1-1).
\end{aligned}$$

Equation (2.27) includes the third line of the second part of equation (2.24).

The last three lines of equation (2.24) do not change from the final form of $P_{OUT}(SW)$ for the case where $0 \leq SW \leq x_1$; that is,

$$\begin{aligned}
&\sum_{z=x_2+1}^{\infty} \begin{bmatrix} +(z-x_2)P(x \geq z-x_2) - \mu P(x \geq z-x_2-1) \\ -(z-x_{Max}-1)P(x \geq z-min\{z,x_{Max}\}) \\ +\mu P(x \geq z-min\{z,x_{Max}\}-1) \end{bmatrix} \\
&= \gamma(1) - \mu\alpha(0) + x_1(x_{Max}+1+\mu) - \left[\frac{x_{Max}(x_{Max}+1)}{2} - \frac{x_2(x_2+1)}{2}\right] \\
&\quad - \gamma(1) + \alpha(1) + \mu\alpha(0) \\
&= x_1(x_{Max}+1+\mu) - \left[\frac{x_{Max}(x_{Max}+1)}{2} - \frac{x_2(x_2+1)}{2}\right] + \alpha(1).
\end{aligned}$$

Putting all the pieces together gives, for $x_1+1 \le SW \le x_2$,

$$P_{OUT}(SW)=\frac{1}{Q_P Q_R}\left\{\begin{array}{l}(x_1+1)(x_2-SW+1)-\gamma(SW+1)+\mu\alpha(SW)\\ +\gamma(SW-x_1)-\mu\alpha(SW-x_1-1)\\ +x_1(x_{Max}+1+\mu)-\left[\frac{x_{Max}(x_{Max}+1)}{2}-\frac{x_2(x_2+1)}{2}\right]\\ +\alpha(1)\end{array}\right\}. \quad (2.28)$$

3. The case of $x_2 < SW \le x_{Max}$

In this case all that remains of equation (2.3) is the third part.

$$P_{OUT}(SW)=\frac{1}{Q_P Q_R}\sum_{z=SW}^{\infty}\left[\begin{array}{l}-(z+1)P(x\ge z+1)+\mu P(x\ge z)\\ +(z-x_1)P(x\ge z-x_1)-\mu P(x\ge z-x_1-1)\\ +(z-x_2)P(x\ge z-x_2)-\mu P(x\ge z-x_2-1)\\ -(z-x_{Max}-1)P(x\ge z-\min\{z,x_{Max}\})\\ +\mu P(x\ge z-\min\{z,x_{Max}\}-1)\end{array}\right]. \quad (2.29)$$

There are no constant terms in equation (2.29). The summation results for the first two lines have been determined above in equations (2.26) and (2.27). All that remains to be derived are the summations for the last three lines of equation (2.29). Similarly to the derivations leading to equations (2.17) and (2.19), we have

$$-\mu\sum_{z=SW}^{\infty}P(x\ge z-x_2-1)=-\mu\sum_{y=SW-x_2-1}^{\infty}P(x\ge y)=-\mu\alpha(SW-x_2-1);$$

$$\sum_{z=SW}^{\infty}(z-x_2)P(x\geq z-x_2)=\sum_{y=SW-x_2}^{\infty}yP(x\geq y)=\gamma(SW-x_2).$$

Two terms still remain to be simplified. These are the last two lines; namely,

$$-\sum_{z=SW}^{\infty}(z-x_{Max}-1)P(x\geq z-min\{z,x_{Max}\});$$

$$\sum_{z=SW}^{\infty}\mu P(x\geq z-min\{z,x_{Max}\}-1).$$

It is useful to separate the summation into two parts; from SW to x_{Max}, and from $x_{Max}+1$ to ∞. The first summations result in

$$\begin{aligned}(x_{Max}+1)\sum_{z=SW}^{x_{Max}}P(x\geq z-min\{z,x_{Max}\}) &= (x_{Max}+1)\sum_{z=SW}^{x_{Max}}P(x\geq 0)\\ &=(x_{Max}+1)(x_{Max}-SW+1);\end{aligned}$$

$$\sum_{z=SW}^{x_{Max}}\mu P(x\geq z-min\{z,x_{Max}\}-1)=\mu\sum_{z=SW}^{x_{Max}}P(x\geq -1)=\mu(x_{Max}-SW+1);$$

and

$$\begin{aligned}-\sum_{z=SW}^{x_{Max}}zP(x\geq z-min\{z,x_{Max}\}) &=-\sum_{z=SW}^{x_{Max}}zP(x\geq 0)=-\sum_{z=SW}^{x_{Max}}z\\ &=-\left[\frac{x_{Max}(x_{Max}+1)}{2}-\frac{(SW-1)SW}{2}\right].\end{aligned}$$

Putting these parts together gives

$$
\begin{aligned}
&-\sum_{z=SW}^{x_{Max}} zP(x \geq z - min\{z, x_{Max}\}) \\
&+(x_{Max}+1)\sum_{z=SW}^{x_{Max}} P(x \geq z - min\{z, x_{Max}\}) \\
&+\sum_{z=SW}^{x_{Max}} \mu P(x \geq z - min\{z, x_{Max}\} - 1) \\
&= (x_{Max}+1+\mu)(x_{Max}-SW+1) - \left[\frac{x_{Max}(x_{Max}+1)}{2} - \frac{(SW-1)SW}{2}\right] .
\end{aligned}
$$

Equation (2.22) provides the results for the summations from $x_{Max}+1$ to ∞. Thus, for the case where $x_2 < SW \leq x_{Max}$ the sum of the last two lines of equation (2.29) is

$$
\begin{aligned}
&-\sum_{z=SW}^{\infty} (z - x_{Max} - 1)P(x \geq z - min\{z, x_{Max}\}) \\
&+\sum_{z=SW}^{\infty} \mu P(x \geq z - min\{z, x_{Max}\} - 1) \\
&= (x_{Max}+1+\mu)(x_{Max}-SW+1) - \left[\frac{x_{Max}(x_{Max}+1)}{2} - \frac{(SW-1)SW}{2}\right] \\
&\quad -\gamma(1)+\alpha(1)+\mu\alpha(0).
\end{aligned}
$$

Therefore, when $x_2 < SW \le x_{Max}$,

$$P_{OUT}(SW) = \frac{1}{Q_P Q_R}\left\{\begin{array}{l} +(x_{Max}+1+\mu)(x_{Max}-SW+1) \\ -\left[\dfrac{x_{Max}(x_{Max}+1)}{2} - \dfrac{(SW-1)SW}{2}\right] \\ -\gamma(SW+1)+\mu\alpha(SW) \\ +\gamma(SW-x_1)-\mu\alpha(SW-x_1-1) \\ +\gamma(SW-x_2)-\mu\alpha(SW-x_2-1) \\ -\gamma(1)+\alpha(1)+\mu\alpha(0) \end{array}\right\}. \qquad (2.30)$$

5. The case of $SW > x_{Max}$

For the case where $SW > x_{Max}$ the first two lines of equation (2.30) are no longer applicable and the last two lines of equation (2.29) are

$$\begin{aligned} &\sum_{z=SW}^{\infty} (-z + x_{Max} + 1) P(x \ge z - min\{z, x_{Max}\}) \\ &\quad = -\sum_{z=SW}^{\infty} (z - x_{Max}) P(x \ge z - x_{Max}) + \sum_{z=SW}^{\infty} P(x \ge z - x_{Max}) \\ &\quad = -\sum_{y=SW-x_{Max}}^{\infty} yP(x \ge y) + \sum_{y=SW-x_{Max}}^{\infty} P(x \ge y) = -\gamma(SW - x_{Max}) + \alpha(SW - x_{Max}); \end{aligned}$$

and

$$\begin{aligned} &\sum_{z=SW}^{\infty} \mu P(x \ge z - min\{z, x_{Max}\} - 1) = \mu \sum_{z=SW}^{\infty} P(x \ge z - x_{Max} - 1) \\ &\quad = \mu \sum_{y=SW-x_{Max}-1}^{\infty} P(x \ge y) = \mu\alpha(SW - x_{Max} - 1). \end{aligned}$$

Therefore, for $SW > x_{Max}$,

$$P_{OUT}(SW) = \frac{1}{Q_P Q_R} \begin{Bmatrix} -\gamma(SW+1) + \mu\alpha(SW) \\ +\gamma(SW - x_1) - \mu\alpha(SW - x_1 - 1) \\ +\gamma(SW - x_2) - \mu\alpha(SW - x_2 - 1) \\ -\gamma(SW - x_{Max}) + \alpha(SW - x_{Max}) \\ +\mu\alpha(SW - x_{Max} - 1) \end{Bmatrix}. \qquad (2.31)$$

D. Summary

This completes the derivations of the equations for $P_{OUT}(SW)$. Equation (2.23) applies when $0 \le SW \le x_1$, equation (2.28) applies for $x_1 < SW \le x_2$, equation (2.30) applies when $x_2 < SW \le x_{Max}$, and equation (2.31) applies for $SW > x_{Max}$. These are the equations which were programmed for the Poisson distribution representing the demand during aggregate lead time for a repairable item.

CHAPTER 3 – DERIVATION OF $B(SW)$ FOR BAKER'S MODEL

A. Introduction

This chapter presents the derivations of the formulas for all of the cases for the expected number of backorders at any instant of time, $B(SW)$, for the Baker model [1] where demand during aggregate lead time is Poisson. The derivations will rely heavily on the derivations in Chapter 2. Indeed, we begin with the case where $0 \le SW \le x_1$ and recall equation (2.12). Instead of merely summing all elements of the net inventory probability distribution provided by Baker (equation (2.2), which the modified form) as we did for $P_{OUT}(SW)$, we must include the number of backorders for each case; that is, $z - SW$ when $z > SW$. However, we will also include the case where $z = SW$ so that we can use the derivations of $P_{OUT}(SW)$ as part of the backorder derivations. We do realize that when $z = SW$ that there are no backorders but it is mathematical convenient to include it anyway. For Baker's model we define $B(SW)$ as

$$B(SW) = \sum_{z=SW}^{\infty} (z - SW)[N(t) = SW - z]. \tag{3.1}$$

Next, we can separate this equation into the following two parts;

$$B(SW) = \sum_{z=SW}^{\infty} z[N(t) = SW - z] - SW \sum_{z=SW}^{\infty} [N(t) = SW - z] \quad . \tag{3.2}$$

The second term is nothing more than the product $SW\,P_{OUT}(SW)$. Therefore we can focus our attention on the derivations associated with the first part. We will denote this part as $B_1(SW)$.

$$B_1(SW) = \sum_{z=SW}^{\infty} z[N(t) = SW - z]. \tag{3.3}$$

Therefore, we can write

$$B(SW) = B_1(SW) - SW\,P_{OUT}(SW). \tag{3.4}$$

B. Derivation of $B_1(SW)$

1. The case of $0 \le SW \le x_1$

For the case of $0 \le SW \le x_1$, when the net inventory probability distribution is given by equation (2.2) we can write $B_1(SW)$ as

$$\begin{aligned} B_1(SW) = {} & \frac{1}{Q_P Q_R} \sum_{z=SW}^{x_1} z[(z+1-\mu) - (z+1)P(x \ge z+1) + \mu P(x \ge z)] \\ & + \frac{1}{Q_P Q_R} \sum_{z=x_1+1}^{x_2} z \begin{bmatrix} (x_1+1) - (z+1)Px \ge z+1) + \mu P(x \ge z) \\ +(z-x_1)P(x \ge z-x_1) - \mu P(x \ge z - x_1 - 1) \end{bmatrix} \\ & + \frac{1}{Q_P Q_R} \sum_{z=x_2+1}^{\infty} z \begin{bmatrix} -(z+1)P(x \ge z+1) + \mu P(x \ge z) \\ +(z-x_1)P(x \ge z-x_1) - \mu P(x \ge z-x_1-1) \\ +(z-x_2)P(x \ge z-x_2) - \mu P(x \ge z-x_2-1) \\ -(z-x_{Max}-1)P(x \ge z - \min\{z, x_{Max}\}) \\ +\mu P(x \ge z - \min\{z, x_{Max}\} - 1) \end{bmatrix}. \end{aligned} \tag{3.5}$$

Next, following the form of equation (2.12), equation (3.5) can be rewritten as

$$B_1(SW)=\frac{1}{Q_PQ_R}\left[\sum_{z=SW}^{x_1} z(z+1-\mu)+\sum_{z=x_1+1}^{x_2} z(x_1+1)\right]$$

$$+\frac{1}{Q_PQ_R}\sum_{z=SW}^{\infty} z[-(z+1)P(x\geq z+1)+\mu P(x\geq z)] \qquad (3.6)$$

$$+\frac{1}{Q_PQ_R}\sum_{z=x_1+1}^{\infty} z[(z-x_1)P(x\geq z-x_1)-\mu P(x\geq z-x_1-1)]$$

$$+\frac{1}{Q_PQ_R}\sum_{z=x_2+1}^{\infty} z\begin{bmatrix}(z-x_2)P(x\geq z-x_2)-\mu P(x\geq z-x_2-1)\\ -(z-x_{Max}-1)P(x\geq z-\min\{z,x_{Max}\})\\ +\mu P(x\geq z-\min\{z,x_{Max}\}-1)\end{bmatrix}.$$

Addressing the constant terms first,

$$\sum_{z=SW}^{x_1} z(z+1-\mu)=\sum_{z=SW}^{x_1} z^2+(1-\mu)\sum_{z=SW}^{x_1} z$$
$$=\left[\frac{x_1(x_1+1)(2x_1+1)}{6}-\frac{SW(SW-1)(2SW-1)}{6}\right] \qquad (3.7)$$
$$+(1-\mu)\left[\frac{x_1(x_1+1)}{2}-\frac{(SW-1)SW}{2}\right] \quad ;$$

$$\sum_{z=x_1+1}^{x_2} z(x_1+1)=(x_1+1)\sum_{z=x_1+1}^{x_2} z=(x_1+1)\left[\frac{x_2(x_2+1)}{2}-\frac{x_1(x_1+1)}{2}\right]. \qquad (3.8)$$

The second line of equation (3.7) is a consequence of the identity

$$\sum_{z=0}^{n} z^2\equiv\frac{n(n+1)(2n+1)}{6} \quad .$$

Adding equations (3.7) and (3.8) and collecting like terms gives:

$$\sum_{z=SW}^{x_1} z(z+1-\mu)+\sum_{z=x_1+1}^{x_2} z(x_1+1)=\left[\frac{x_1(x_1+1)(2x_1+1)}{6}-\frac{SW(SW-1)(2SW-1)}{6}\right]$$
$$+x_1\left[\frac{x_2(x_2+1)}{2}-\frac{x_1(x_1+1)}{2}\right]$$
$$-\mu\left[\frac{x_1(x_1+1)}{2}-\frac{SW(SW-1)}{2}\right]$$
$$+\frac{x_2(x_2+1)}{2}-\frac{SW(SW-1)}{2} \quad . \qquad (3.9)$$

We next need to develop the parts involving the probabilities. The third line of equation (3.6) is first.

$$\sum_{z=SW}^{\infty} z[-(z+1)P(x\geq z+1)+\mu P(x\geq z)]$$
$$=-\sum_{z=SW}^{\infty} z^2P(x\geq z+1)-\sum_{z=SW}^{\infty} zP(x\geq z+1)+\mu\sum_{z=SW}^{\infty} zP(x\geq z). \qquad (3.10)$$

Looking at the first term on the right-hand side, let $y=z+1$. Then,

$$z^2=(y-1)^2=y^2-2y+1 \ .$$

Also when $z=SW$ then $y=SW+1$. This change of variable gives

$$-\sum_{z=SW}^{\infty} z^2P(x\geq z+1)=-\sum_{y=SW+1}^{\infty}(y^2-2y+1)P(x\geq y)$$
$$=-\sum_{y=SW+1}^{\infty} y^2P(x\geq y)+2\sum_{y=SW+1}^{\infty} yP(x\geq y)-\sum_{y=SW+1}^{\infty} P(x\geq y). \qquad (3.11)$$

Next we define $\delta(u)\equiv\sum_{y=u}^{\infty} y^2P(x\geq y)$ and Identity 9 of Appendix 3 of Reference{3}

provides the following result:

$$\delta(u) \equiv \sum_{y=u}^{\infty} y^2 P(x \geq y) = \left[-\frac{u^3}{3} + \frac{u^2}{2} - \frac{u}{6} \right] P(x \geq u) + \mu P(x \geq u-1)$$
$$+ \frac{3\mu^2}{2} P(x \geq u-2) + \frac{\mu^3}{3} P(x \geq u-3) . \tag{3.12}$$

The definitions of $\alpha(u)$ and $\gamma(u)$ were given in Chapter 2 by equations (2.13) and (2.16), respectively. Thus, the second line of equation (3.11) in final reduced form is:

$$-\sum_{z=SW}^{\infty} z^2 P(x \geq z+1) = -\delta(SW+1) + 2\gamma(SW+1) - \alpha(SW+1) . \tag{3.13}$$

The second term of equation (3.10) is similarly analyzed. We get

$$\begin{aligned} -\sum_{z=SW}^{\infty} zP(x \geq z+1) &= -\sum_{y=SW+1}^{\infty} (y-1)P(x \geq y) \\ &= -\sum_{y=SW+1}^{\infty} yP(x \geq y) + \sum_{y=SW+1}^{\infty} P(x \geq SW) \qquad (3.14) \\ &= -\gamma(SW+1) + \alpha(SW+1) . \end{aligned}$$

The third term of equation (3.10) is next. No transformation of variable is needed.

$$\mu \sum_{z=SW}^{\infty} zP(x \geq z) = \mu\gamma(SW) . \tag{3.15}$$

The final form for equation (3.10) is the sum of the right-hand sides of equations (3.11), (3.12), and (3.13):

$$\sum_{z=SW}^{\infty} z[-(z+1)P(x \geq z+1) + \mu P(x \geq z)] = -\delta(SW+1) + \gamma(SW+1) + \mu\gamma(SW). \quad (3.16)$$

Next we look at the third line of equation (3.6). It can be subdivided into three parts.

$$\begin{aligned}
&\sum_{z=x_1+1}^{\infty} z[(z-x_1)P(x \geq z-x_1) - \mu P(x \geq z-x_1-1)] \\
&\quad = \sum_{z=x_1+1}^{\infty} z^2 P(x \geq z-x_1) \\
&\quad - x_1 \sum_{z=x_1+1}^{\infty} zP(x \geq z-x_1) \\
&\quad - \mu \sum_{z=x_1+1}^{\infty} zP(x \geq z-x_1-1).
\end{aligned} \quad (3.17)$$

The first term on the right-hand side can be reduced as follows:

$$\begin{aligned}
\sum_{z=x_1+1}^{\infty} z^2 P(x \geq z-x_1) &= \sum_{y=1}^{\infty} (y+x_1)^2 P(x \geq y) \\
&= \sum_{y=1}^{\infty} y^2 P(x \geq y) + 2x_1 \sum_{y=1}^{\infty} yP(x \geq y) + x_1{}^2 \sum_{y=1}^{\infty} P(x \geq y) \quad (3.18) \\
&= \delta(1) + 2x_1\gamma(1) + x_1{}^2\alpha(1).
\end{aligned}$$

The second term is

$$-x_1 \sum_{z=x_1+1}^{\infty} zP(x \geq z-x_1) = -x_1 \sum_{y=1}^{\infty} (y+x_1)P(x \geq y) = -x_1\gamma(1) - x_1{}^2\alpha(1). \quad (3.19)$$

The third term is

$$\begin{aligned}
-\mu \sum_{z=x_1+1}^{\infty} zP(x \geq z-x_1-1) &= -\mu \sum_{y=0}^{\infty} (y+x_1+1)P(x \geq y) \\
&= -\mu\gamma(0) - \mu(x_1+1)\alpha(0).
\end{aligned} \quad (3.20)$$

Summing equations (3.18), (3.19), and (3.20) results in equation (3.17) becoming

$$\sum_{z=x_1+1}^{\infty} z\left[(z-x_1)P(x \geq z-x_1) - \mu P(x \geq z-x_1-1)\right]$$
$$= \delta(1) + x_1\gamma(1) - \mu\gamma(0) - \mu(x_1+1)\alpha(0). \tag{3.21}$$

The next term to consider is the first line of the last part of equation (3.6).

$$\begin{aligned}
&\sum_{z=x_2+1}^{\infty} z\left[(z-x_2)P(x \geq z-x_2) - \mu P(x \geq z-x_2-1)\right] \\
&\quad = \sum_{z=x_2+1}^{\infty} z^2 P(x \geq z-x_2) \\
&\qquad - x_2 \sum_{z=x_2+1}^{\infty} zP(x \geq z-x_2) \\
&\qquad - \mu \sum_{z=x_2+1}^{\infty} zP(x \geq z-x_2-1).
\end{aligned} \tag{3.22}$$

Following the steps for equation (3.18) we can write the first term on the right-hand side of equation (3.22) as

$$\begin{aligned}
\sum_{z=x_2+1}^{\infty} z^2 P(x \geq z-x_2) &= \sum_{y=1}^{\infty} (y+x_2)^2 P(x \geq y) \\
&= \sum_{y=1}^{\infty} y^2 P(x \geq y) + 2x_2 \sum_{y=1}^{\infty} yP(x \geq y) + x_2{}^2 \sum_{y=1}^{\infty} P(x \geq y) \\
&= \delta(1) + 2x_2\gamma(1) + x_2{}^2\alpha(1).
\end{aligned} \tag{3.23}$$

Following the steps of equation (3.19) we can write the second right-hand term of equation (3.22) as

$$-x_2 \sum_{z=x_2+1}^{\infty} zP(x \geq z-x_2) = -x_2 \sum_{y=1}^{\infty} (y+x_2)P(x \geq y) = -x_2\gamma(1) - {x_2}^2\alpha(1). \qquad (3.24)$$

Finally, following equation (3.19) we can write the third right-hand term of equation (3.22) as

$$\begin{aligned} -\mu \sum_{z=x_2+1}^{\infty} zP(x \geq z-x_2-1) &= -\mu \sum_{y=0}^{\infty} (y+x_2+1)P(x \geq y) \\ &= -\mu\gamma(0) - \mu(x_2+1)\alpha(0). \end{aligned} \qquad (3.25)$$

Summing the right-hand sides of equations (3.23), (3.24), and (3.25) results in equation (3.22) being reduced to

$$\begin{aligned} &\sum_{z=x_2+1}^{\infty} z\left[(z-x_2)P(x \geq z-x_2) - \mu P(x \geq z-x_2-1)\right] \\ &\quad = \delta(1) + x_2\gamma(1) - \mu\gamma(0) - \mu(x_2+1)\alpha(0). \end{aligned} \qquad (3.26)$$

The last two lines of equation (3.6) are

$$\sum_{z=x_2+1}^{\infty} -z\left[(z-x_{Max}-1)P(x \geq z-\min(z,x_{Max})\right] + \mu \sum_{z=x_2+1}^{\infty} zP(x \geq z-\min\{z,x_{Max}\}-1).$$

As in the $P_{OUT}(SW)$ derivations, we subdivide this summation into two parts; from x_2+1 *to* x_{Max} and from $x_{Max}+1$ to ∞. In the first part of the summation interval we have

$$\begin{aligned} &\sum_{z=x_2+1}^{x_{Max}} -z\left[(z-x_{Max}-1)P(x \geq z-\min(z,x_{Max})\right] \\ &\quad = \sum_{z=x_2+1}^{x_{Max}} -z^2P(x \geq 0) + (x_{Max}+1) \sum_{z=x_2+1}^{x_{Max}} zP(x \geq 0), \end{aligned}$$

since $z - Min\{z, x_{Max}\} = z - z = 0$. Likewise,

$$\mu \sum_{z=x_2+1}^{x_{Max}} zP(x \geq z - \min\{z, x_{Max}\} - 1) = \mu \sum_{z=x_2+1}^{x_{Max}} zP(x \geq -1).$$

Because $P(x \geq 0) = 1.0$ and $P(x \geq -1) = 1.0$ for the Poisson distribution these equations are reduced to

$$\begin{aligned}
&\sum_{z=x_2+1}^{\infty} -z[(z - x_{Max} - 1)P(x \geq z - \min(z, x_{Max})] \\
&\qquad = \sum_{z=x_2+1}^{x_{Max}} -z^2 + (x_{Max} + 1) \sum_{z=x_2+1}^{x_{Max}} z \\
&\qquad = -\left[\frac{x_{Max}(x_{Max}+1)(2x_{Max}+1)}{6} - \frac{x_2(x_2+1)(2x_2+1)}{6}\right] \\
&\qquad\quad + (x_{Max}+1)\left[\frac{x_{Max}(x_{Max}+1)}{2} - \frac{x_2(x_2+1)}{2}\right],
\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}
&\mu \sum_{z=x_2+1}^{x_{Max}} zP(x \geq z - \min\{z, x_{Max}\} - 1) = \mu \sum_{z=x_2+1}^{x_{Max}} z \\
&\qquad = \mu\left[\frac{x_{Max}(x_{Max}+1)}{2} - \frac{x_2(x_2+1)}{2}\right].
\end{aligned} \tag{3.28}$$

Combining equations (3.27) and (3.28),

$$\sum_{z=x_2+1}^{x_{Max}} -z\left[(z-x_{Max}-1)P(x \geq z-\min\{z,x_{Max}\})\right]$$

$$+\mu \sum_{z=x_2+1}^{x_{Max}} zP(x \geq z-\min\{z,x_{Max}\}-1) \tag{3.29}$$

$$=-\left[\frac{x_{Max}(x_{Max}+1)(2x_{Max}+1)}{6}-\frac{x_2(x_2+1)(2x_2+1)}{6}\right]$$

$$+(x_{Max}+1+\mu)\left[\frac{x_{Max}(x_{Max}+1)}{2}-\frac{x_2(x_2+1)}{2}\right].$$

The remaining part of the summation is for $x_{Max}+1$ to ∞.

$$\sum_{z=x_{Max}+1}^{\infty} -z\left[(z-x_{Max}-1)P(x \geq z-\min\{z,x_{Max}\})\right]$$

$$+\mu \sum_{z=x_{Max}+1}^{\infty} zP(x \geq z-\min\{z,x_{Max}\}-1) \tag{3.30}$$

$$=\sum_{z=x_{Max}+1}^{\infty} -z\left[(z-x_{Max}-1)P(x \geq z-x_{Max}\right]$$

$$+\mu \sum_{z=x_{Max}+1}^{\infty} zP(x \geq z-x_{Max}-1).$$

The first line of the right-hand side of this equation can be reduced as follows:

$$\sum_{z=x_{Max}+1}^{\infty} -z[(z-x_{Max}-1)P(x \geq z-x_{Max}]$$

$$\begin{aligned}
&= -\sum_{z=x_{Max}+1}^{\infty} z^2 P(x \geq z-x_{Max}) + (x_{Max}+1)\sum_{z=x_{Max}+1}^{\infty} zP(x \geq z-x_{Max}) \\
&= -\sum_{y=1}^{\infty}(y+x_{Max})^2 P(x \geq y) + (x_{Max}+1)\sum_{y=1}^{\infty}(y+x_{Max})P(x \geq y) \\
&= -\sum_{y=1}^{\infty} y^2 P(x \geq y) - 2x_{Max}\sum_{y=1}^{\infty} yP(x \geq y) - x_{Max}^2 \sum_{y=1}^{\infty} P(x \geq y) \\
&\quad + (x_{Max}+1)\sum_{y=1}^{\infty} yP(x \geq y) + x_{Max}(x_{Max}+1)\sum_{y=1}^{\infty} P(x \geq y) \\
&= -\delta(1) - 2x_{Max}\gamma(1) - x_{Max}^2\alpha(1) + (x_{Max}+1)\gamma(1) + x_{Max}(x_{Max}+1)\alpha(1) \\
&= -\delta(1) - (x_{Max}-1)\gamma(1) + x_{Max}\alpha(1).
\end{aligned} \tag{3.31}$$

The last line of equation (3.30) is reduced to

$$\begin{aligned}
\mu \sum_{z=x_{Max}+1}^{\infty} zP(x \geq z-x_{Max}-1) &= \mu \sum_{y=0}^{\infty}(y+x_{Max}+1)P(x \geq y) \\
&= \mu\gamma(0) + \mu(x_{Max}+1)\alpha(0).
\end{aligned} \tag{3.32}$$

This completes the derivations for $B_1(SW)$ for the case of $0 \leq SW \leq x_1$.

Equation (3.6) now has the following form:

$$B_1(SW)=\frac{1}{Q_PQ_R}\left\{\begin{array}{l}\left[\frac{x_1(x_1+1)(2x_1+1)}{6}-\frac{SW(SW-1)(2SW-1)}{6}\right] \\ +x_1\left[\frac{x_2(x_2+1)}{2}-\frac{x_1(x_1+1)}{2}\right] \\ -\mu\left[\frac{x_1(x_1+1)}{2}-\frac{SW(SW-1)}{2}\right] \\ +\frac{x_2(x_2+1)}{2}-\frac{SW(SW-1)}{2} \\ -\delta(SW+1)+\gamma(SW+1)+\mu\gamma(SW) \\ \\ +\delta(1)+x_1\gamma(1)-\mu\gamma(0)-\mu(x_1+1)\alpha(0) \\ \\ +\delta(1)+x_2\gamma(1)-\mu\gamma(0)-\mu(x_2+1)\alpha(0) \\ \\ -\left[\frac{x_{Max}(x_{Max}+1)(2x_{Max}+1)}{6}-\frac{x_2(x_2+1)(2x_2+1)}{6}\right] \\ +(x_{Max}+1+\mu)\left[\frac{x_{Max}(x_{Max}+1)}{2}-\frac{x_2(x_2+1)}{2}\right] \\ \\ -\delta(1)-(x_{Max}-1)\gamma(1)+x_{Max}\alpha(1) \\ \\ +\mu\gamma(0)+\mu(x_{Max}+1)\alpha(0)\end{array}\right\}. \quad (3.33)$$

Note that there are terms which can be cancelled; such as $\delta(1)$'s in the seventh and tenth lines and the $\mu\gamma(0)$ terms in the seventh and eleventh lines. All of the $\gamma(1)$ terms can be collected and reduced.

$$\begin{aligned} x_1\gamma(1)+x_2\gamma(1)-(x_{Max}-1)\gamma(1) &= [x_1-(x_{Max}-x_2)+1]\gamma(1) \\ &= [x_1-x_1+1]\gamma(1)=\gamma(1). \end{aligned}$$

Similarly, the $\alpha(0)$ terms can be collected.

$$
\begin{aligned}
&-\mu(x_1+1)\alpha(0)-\mu(x_2+1)\alpha(0)+\mu(x_{Max}+1)\alpha(0)\\
&\quad=\left[-x_1-x_2+x_{Max}-1-1+1\right]\mu\alpha(0)\\
&\quad=\left[-x_1+x_1-1\right]\mu\alpha(0)=-\mu\alpha(0)\,.
\end{aligned}
$$

This allows consolidation of the sixth, seventh, tenth and eleventh lines into

$$
\begin{aligned}
&\delta(1)+x_1\gamma(1)-\mu\gamma(0)-\mu(x_1+1)\alpha(0)\\
&\quad+\delta(1)+x_2\gamma(1)-\mu\gamma(0)-\mu(x_2+1)\alpha(0)\\
&\quad-\delta(1)-(x_{Max}-1)\gamma(1)+x_{Max}\alpha(1)\\
&\quad+\mu\gamma(0)+\mu(x_{Max}+1)\alpha(0)\\
&=\delta(1)+\gamma(1)+x_{Max}\alpha(1)-\mu\gamma(0)-\mu\alpha(0)\,.
\end{aligned}
$$

Equation (3.33) is then reduced to

$$
B_1(SW)=\frac{1}{Q_PQ_R}\left\{\begin{aligned}
&\left[\frac{x_1(x_1+1)(2x_1+1)}{6}-\frac{SW(SW-1)(2SW-1)}{6}\right]\\
&+x_1\left[\frac{x_2(x_2+1)}{2}-\frac{x_1(x_1+1)}{2}\right]\\
&-\mu\left[\frac{x_1(x_1+1)}{2}-\frac{SW(SW-1)}{2}\right]\\
&+\frac{x_2(x_2+1)}{2}-\frac{SW(SW-1)}{2}\\
&-\delta(SW+1)+\gamma(SW+1)+\mu\gamma(SW)\\
&-\left[\frac{x_{Max}(x_{Max}+1)(2x_{Max}+1)}{6}-\frac{x_2(x_2+1)(2x_2+1)}{6}\right]\\
&+(x_{Max}+1+\mu)\left[\frac{x_{Max}(x_{Max}+1)}{2}-\frac{x_2(x_2+1)}{2}\right]\\
&+\delta(1)+\gamma(1)+x_{Max}\alpha(1)-\mu\gamma(0)-\mu\alpha(0)
\end{aligned}\right\}. \qquad (3.34)
$$

However, from a programming point of view little is gained by using equation (3.34). In addition, as the forms of $B_1(SW)$ for the other cases are

developed below it is useful to compare them with equation (3.33) rather than equation (3.34).

2. The case of $x_1 < SW \le x_2$.

Equation (3.6) is now reduced to

$$\begin{aligned} B_1(SW) = {} & \frac{1}{Q_P Q_R} \sum_{z=SW}^{x_2} z(x_1+1) \\ & + \frac{1}{Q_P Q_R} \sum_{z=SW}^{\infty} z[-(z+1)P(x \ge z+1) + \mu P(x \ge z)] \\ & + \frac{1}{Q_P Q_R} \sum_{z=SW}^{\infty} z[(z-x_1)P(x \ge z-x_1) - \mu P(x \ge z-x_1-1)] \\ & + \frac{1}{Q_P Q_R} \sum_{z=x_2+1}^{\infty} z \begin{bmatrix} (z-x_2)P(x \ge z-x_2) - \mu P(x \ge z-x_2-1) \\ -(z-x_{Max}-1)P(x \ge z-\min\{z,x_{Max}\}) \\ +\mu P(x \ge z-\min\{z,x_{Max}\}-1) \end{bmatrix}. \end{aligned} \tag{3.35}$$

The constant term is easily reduced.

$$\sum_{z=SW}^{x_2} z(x_1+1) = (x_1+1)\sum_{z=SW}^{x_2} z = (x_1+1)\left[\frac{x_2(x_2+1)}{2} - \frac{(SW-1)SW}{2}\right]. \tag{3.36}$$

The second and fourth through sixth lines of equation (3.35) remain the same for this case as for $0 \le SW \le x_1$. The third line changes. Following the equation (3.17) analysis, we have

$$
\begin{aligned}
&\sum_{z=SW}^{\infty} z\left[(z-x_1)P(x \geq z-x_1) - \mu P(x \geq z-x_1-1)\right] \\
&\quad = \sum_{z=SW}^{\infty} z^2 P(x \geq z-x_1) \\
&\quad - x_1 \sum_{z=SW}^{\infty} zP(x \geq z-x_1) \\
&\quad - \mu \sum_{z=SW}^{\infty} zP(x \geq z-x_1-1).
\end{aligned}
\tag{3.37}
$$

The first term of equation (3.37) is reduced as follows:

$$
\begin{aligned}
\sum_{z=SW}^{\infty} z^2 P(x \geq z-x_1) &= \sum_{y=SW-x_1}^{\infty} (y+x_1)^2 P(x \geq y) \\
&= \sum_{y=SW-x_1}^{\infty} y^2 P(x \geq y) + 2x_1 \sum_{y=SW-x_1}^{\infty} yP(x \geq y) + x_1^2 \sum_{y=SW-x_1}^{\infty} P(x \geq y) \\
&= \delta(SW-x_1) + 2x_1\gamma(SW-x_1) + x_1^2\alpha(SW-x_1).
\end{aligned}
\tag{3.38}
$$

The second term becomes:

$$
\begin{aligned}
-x_1 \sum_{z=SW}^{\infty} zP(x \geq z-x_1) &= -x_1 \sum_{y=SW-x_1}^{\infty} (y+x_1)P(x \geq y) \\
&= -x_1\gamma(SW-x_1) - x_1^2\alpha(SW-x_1).
\end{aligned}
\tag{3.39}
$$

And, the third term is:

$$
\begin{aligned}
-\mu \sum_{z=SW}^{\infty} zP(x \geq z-x_1-1) &= -\mu \sum_{y=SW-x_1-1}^{\infty} (y+x_1+1)P(x \geq y) \\
&= -\mu\gamma(SW-x_1-1) - \mu(x_1+1)\alpha(SW-x_1-1).
\end{aligned}
\tag{3.40}
$$

Summing the results of equations (3.38), (3.39), and (3.40) gives equation (3.37) in the following reduced form:

$$\begin{aligned}\sum_{z=SW}^{\infty} z[(z-x_1)P(x \geq z-x_1) - \mu P(x \geq z-x_1-1)] \\ = \delta(SW-x_1) + x_1\gamma(SW-x_1) \\ - \mu\gamma(SW-x_1-1) - \mu(x_1+1)\alpha(SW-x_1-1)\,.\end{aligned} \tag{3.41}$$

The form for $B_1(SW)$ is therefore given by equation (3.42) for this case.

$$B_1(SW) = \frac{1}{Q_P Q_R}\left\{\begin{array}{l} (x_1+1)\left[\dfrac{x_2(x_2+1)}{2} - \dfrac{(SW-1)SW}{2}\right] \\ -\delta(SW+1) + \gamma(SW+1) + \mu\gamma(SW) \\ +\delta(SW-x_1) + x_1\gamma(SW-x_1) \\ -\mu\gamma(SW-x_1-1) - \mu(x_1+1)\alpha(SW-x_1-1) \\ +\delta(1) + x_2\gamma(1) - \mu\gamma(0) - \mu(x_2+1)\alpha(0) \\ -\left[\dfrac{x_{Max}(x_{Max}+1)(2x_{Max}+1)}{6} - \dfrac{x_2(x_2+1)(2x_2+1)}{6}\right] \\ +(x_{Max}+1+\mu)\left[\dfrac{x_{Max}(x_{Max}+1)}{2} - \dfrac{x_2(x_2+1)}{2}\right] \\ -\delta(1) - (x_{Max}-1)\gamma(1) + x_{Max}\alpha(1) \\ +\mu\gamma(0) + \mu(x_{Max}+1)\alpha(0) \end{array}\right\}. \tag{3.42}$$

Examination of equation (3.42) shows that the $\delta(1)$ and $\mu\gamma(0)$ terms will cancel. In addition, the $\gamma(1)$ and $\alpha(0)$ terms can be combined:

$$-\mu(x_2+1)\alpha(0)+\mu(x_{Max}+1)\alpha(0)=\mu(x_{Max}-x_2)\alpha(0)=\mu x_1\alpha(0);$$

$$x_2\gamma(1)-(x_{Max}-1)\gamma(1)=\left[1-(x_{Max}-x_2)\right]\gamma(1)=(1-x_1)\gamma(1).$$

Equation (3.42) can then be somewhat reduced to

$$B_1(SW)=\frac{1}{Q_P Q_R}\left\{\begin{array}{l}(x_1+1)\left[\frac{x_2(x_2+1)}{2}-\frac{(SW-1)SW}{2}\right]\\ -\delta(SW+1)+\gamma(SW+1)+\mu\gamma(SW)\\ +\delta(SW-x_1)+x_1\gamma(SW-x_1)\\ -\mu\gamma(SW-x_1-1)-\mu(x_1+1)\alpha(SW-x_1-1)\\ +(1-x_1)\gamma(1)+\mu x_1\alpha(0)+x_{Max}\alpha(1)\\ -\left[\frac{x_{Max}(x_{Max}+1)(2x_{Max}+1)}{6}-\frac{x_2(x_2+1)(2x_2+1)}{6}\right]\\ +(x_{Max}+1+\mu)\left[\frac{x_{Max}(x_{Max}+1)}{2}-\frac{x_2(x_2+1)}{2}\right]\end{array}\right\}. \quad (3.43)$$

2. The case of $x_2 < SW \leq x_{Max}$

In this case we need to consider only the third part of equation (3.5).

Now $B_1(SW)$ has the form:

$$B_1(SW) = \frac{1}{Q_P Q_R} \sum_{z=SW}^{\infty} z \begin{bmatrix} -(z+1)P(x \geq z+1) + \mu P(x \geq z) \\ +(z-x_1)P(x \geq z-x_1) - \mu P(x \geq z-x_1-1) \\ +(z-x_2)P(x \geq z-x_2) - \mu P(x \geq z-x_2-1) \\ -(z-x_{Max}-1)P(x \geq z-\min\{z, x_{Max}\}) \\ +\mu P(x \geq z-\min\{z, x_{Max}\}-1) \end{bmatrix}. \quad (3.44)$$

The constant terms from the top lines of equations (3.33) and (3.42) are no longer present. The final form of the first line of equation (3.44) was provided by equation (3.16). The final form of the second line was given by equation (3.41).

The derivation steps for equation (3.41) provide a short cut for the derivation of the final form of the third line of equation (3.44). When x_2 replaces x_1 in equation (3.38) we get

$$\sum_{z=SW}^{\infty} z^2 P(x \geq z-x_2) = \sum_{y=SW-x_2}^{\infty} (y+x_2)^2 P(x \geq y)$$

$$= \sum_{y=SW-x_2}^{\infty} y^2 P(x \geq y) + 2x_2 \sum_{y=SW-x_2}^{\infty} y P(x \geq y) + x_2^2 \sum_{y=SW-x_2}^{\infty} P(x \geq y)$$

$$= \delta(SW-x_2) + 2x_2\gamma(SW-x_2) + x_2^2\alpha(SW-x_2).$$

Similarly, equation (3.39) yields

$$-x_2 \sum_{z=SW}^{\infty} zP(x \geq z-x_2) = -x_2 \sum_{y=SW-x_2}^{\infty} (y+x_2)P(x \geq y)$$

$$= -x_2\gamma(SW-x_2) - x_2^2\alpha(SW-x_2).$$

And, finally, equation (3.40) gives

$$-\mu \sum_{z=SW}^{\infty} zP(x \ge z - x_2 - 1) = -\mu \sum_{y=SW-x_2-1}^{\infty} (y + x_2 + 1)P(x \ge y)$$
$$= -\mu\gamma(SW - x_2 - 1) - \mu(x_2 + 1)\alpha(SW - x_2 - 1).$$

Then, combining these results,

$$\sum_{z=SW}^{\infty} z[(z - x_2)P(x \ge z - x_2) - \mu P(x \ge z - x_2 - 1)]$$
$$= \delta(SW - x_2) + x_2\gamma(SW - x_2) \tag{3.45}$$
$$- \mu\gamma(SW - x_2 - 1) - \mu(x_2 + 1)\alpha(SW - x_2 - 1).$$

The last two lines of equation (3.44) need to be analyzed for the two intervals $x_2 + 1 \le SW \le x_{Max}$ and $SW \ge x_{Max} + 1$. The last two lines for the first interval are split into two summations; those from SW to x_{Max} and from $x_{Max} + 1$ to ∞. From SW to x_{Max}:

$$\sum_{z=SW}^{x_{Max}} -z[(z - x_{Max} - 1)P(x \ge z - \min(z, x_{Max})]$$
$$+ \mu \sum_{z=SW}^{\infty} zP(x \ge z - \min\{z, x_{Max}\} - 1)$$
$$= \sum_{z=SW}^{x_{Max}} -z^2 P(x \ge 0) + (x_{Max} + 1) \sum_{z=SW}^{x_{Max}} zP(x \ge 0) + \mu \sum_{z=SW}^{x_{Max}} zP(x \ge -1) \tag{3.46}$$
$$= -\left[\frac{x_{Max}(x_{Max} + 1)(2x_{Max} + 1)}{6} - \frac{SW(SW - 1)(2SW - 1)}{6}\right]$$
$$+ (x_{Max} + 1 + \mu)\left[\frac{x_{Max}(x_{Max} + 1)}{2} - \frac{SW(SW - 1)}{2}\right].$$

Equations (3.31) and (3.32) provide the remainder of the summation from $x_{Max}+1$ to ∞. Therefore, when $x_2 < SW \le x_{Max}$, $B_1(SW)$ is

$$B_1(SW) = \frac{1}{Q_P Q_R}\left\{\begin{array}{l} -\delta(SW+1)+\gamma(SW+1)+\mu\gamma(SW) \\ +\delta(SW-x_1)+x_1\gamma(SW-x_1)-\mu\gamma(SW-x_1-1) \\ -\mu(x_1+1)\alpha(SW-x_1-1) \\ +\delta(SW-x_2)+x_2\gamma(SW-x_2)-\mu\gamma(SW-x_2-1) \\ -\mu(x_2+1)\alpha(SW-x_2-1) \\ -\left[\frac{x_{Max}(x_{Max}+1)(2x_{Max}+1)}{6}-\frac{SW(SW-1)(2SW-1)}{6}\right] \\ +(x_{Max}+1+\mu)\left[\frac{x_{Max}(x_{Max}+1)}{2}-\frac{SW(SW-1)}{2}\right] \\ -\delta(1)-(x_{Max}-1)\gamma(1)+x_{Max}\alpha(1) \\ +\mu\gamma(0)+\mu(x_{Max}+1)\alpha(0) \end{array}\right\}. \quad (3.47)$$

4. The case of $SW > x_{Max}$

When $SW > x_{Max}$, $B_1(SW)$ is simplified even further because the constant terms of equation (3.47) which are the fourth and fifth lines are eliminated. The last line now involves SW. As a consequence, the lower bound on the summation in equation (3.31) is now SW instead of $x_{Max}+1$ as shown in equation (3.48)

$$
\begin{aligned}
&\sum_{z=SW}^{\infty} -z\left[(z-x_{Max}-1)P(x \geq z-\min\{z,x_{Max}\})\right] \\
&\quad + \mu \sum_{z=SW}^{\infty} zP(x \geq z-\min\{z,x_{Max}\}-1) \\
&= \sum_{z=SW}^{\infty} -z\left[(z-x_{Max}-1)P(x \geq z-x_{Max})\right] \\
&\quad + \mu \sum_{z=SW}^{\infty} zP(x \geq z-x_{Max}-1) \quad .
\end{aligned}
\tag{3.48}
$$

Then, following the same steps as for equation (3.31), the first line of equation (3.48) is reduced as follows;

$$
\begin{aligned}
&\sum_{z=SW}^{\infty} -z\left[(z-x_{Max}-1)P(x \geq z-x_{Max})\right] \\
&= -\sum_{z=SW}^{\infty} z^2P(x \geq z-x_{Max}) + (x_{Max}+1)\sum_{z=SW}^{\infty} zP(x \geq z-x_{Max}) \\
&= -\sum_{y=SW-x_{Max}}^{\infty} (y+x_{Max})^2P(x \geq y) \\
&\quad + (x_{Max}+1)\sum_{y=SW-x_{Max}}^{\infty} (y+x_{Max})P(x \geq y) \\
&= -\sum_{y=SW-x_{Max}}^{\infty} y^2P(x \geq y) - 2x_{Max}\sum_{y=SW-x_{Max}}^{\infty} yP(x \geq y) \\
&\quad - x_{Max}^2\sum_{y=SW-x_{Max}}^{\infty} P(x \geq y) + (x_{Max}+1)\sum_{y=SW-x_{Max}}^{\infty} yP(x \geq y) \\
&\quad + x_{Max}(x_{Max}+1)\sum_{y=SW-x_{Max}}^{\infty} P(x \geq y)
\end{aligned}
\tag{3.49}
$$

$$
\begin{aligned}
&= -\delta(SW-x_{Max}) - 2x_{Max}\gamma(SW-x_{Max}) - x_{Max}^2\alpha(SW-x_{Max}) \\
&\quad + (x_{Max}+1)\gamma(SW-x_{Max}) + x_{Max}(x_{Max}+1)\alpha(SW-x_{Max}) \\
&= -\delta(SW-x_{Max}) - (x_{Max}-1)\gamma(SW-x_{Max}) + x_{Max}\alpha(SW-x_{Max}) .
\end{aligned}
$$

Equation (3.32) provides the structure for the second line of equation(3.48).

$$
\begin{aligned}
&\mu \sum_{z=SW}^{\infty} zP(x \geq z - x_{Max} - 1) \\
&\qquad = \mu \sum_{y=SW-x_{Max}-1}^{\infty} (y + x_{Max} + 1)P(x \geq y) \\
&\qquad = \mu\gamma(SW - x_{Max} - 1) + \mu(x_{Max} + 1)\alpha(SW - x_{Max} - 1).
\end{aligned}
\tag{3.50}
$$

The final form for $B_1(SW)$ when $SW > x_{Max}$ is

$$
B_1(SW) = \frac{1}{Q_P Q_R}\left\{
\begin{array}{l}
-\delta(SW+1) + \gamma(SW+1) + \mu\gamma(SW) \\
+\delta(SW - x_1) + x_1\gamma(SW - x_1) - \mu\gamma(SW - x_1 - 1) \\
-\mu(x_1 + 1)\alpha(SW - x_1 - 1) \\
+\delta(SW - x_2) + x_2\gamma(SW - x_2) - \mu\gamma(SW - x_2 - 1) \\
-\mu(x_2 + 1)\alpha(SW - x_2 - 1) \\
-\delta(SW - x_{Max}) - (x_{Max} - 1)\gamma(SW - x_{Max}) \\
+x_{Max}\alpha(SW - x_{Max}) + \mu\gamma(SW - x_{Max} - 1) \\
+\mu(x_{Max} + 1)\alpha(SW - x_{Max} - 1)
\end{array}
\right\}.
\tag{3.51}
$$

C. Summary

This completes the derivations for $B_1(SW)$. In summary, for $B_1(SW)$, when $0 \leq SW \leq x_1$ we use equation (3.33) or (3.34); when $x_1 < SW \leq x_2$ we use equation (3.42) or (3.43); when $x_2 < SW \leq x_{Max}$ we use equation (3.47); when $SW > x_{Max}$ we use equation (3.51). Finally, to get $B(SW)$ we use equation (3.4) and the corresponding formulas for $P_{OUT}(SW)$ from Chapter 2.

CHAPTER 4 – $P_{OUT}(SW)$ DERIVATION FOR THE NORMAL MODIFICATION

A. Introduction

The derivations presented in this chapter follow those of Chapter 2. However, the assumptions of discreteness for Inventory Position and Net Inventory are no longer used. As a consequence, we will first present the formulas for the Inventory Position and Net Inventory.

B. Inventory Position Distribution

The first step is to redefine the probability distribution for the Inventory Position. The Inventory Position distribution for the discrete demand model was the convolution of two discrete uniform distributions (see Baker [1], pp. 16, 17). Then, using the logic of Hadley and Whitin [3] in their Chapter 4 (page 192) we now have a distribution which is the convolution of two continuous uniform distributions; namely,

$$\begin{aligned} f(x_P) &= \frac{1}{Q_P} \quad \mathit{for}\ 0 \le x_P \le Q_P; \\ f(x_R) &= \frac{1}{Q_R} \quad \mathit{for}\ 0 \le x_R \le Q_R; \end{aligned} \tag{4.1}$$

where x_P is the number of attritions in the procurement batching process at time t and x_R is the number of carcasses in the repair batching process at time t. Letting $x = x_P + x_R$, it follows that if SW is the maximum Inventory Position and I(t) is the Inventory Position at time t, then $I(t) = SW - x$.

The distribution of $I(t)$ is then

$$f(I(t)=SW-x)=\begin{cases}\dfrac{x}{Q_PQ_R} & for \quad 0\le x<x_1\\[2ex] \dfrac{Min(Q_P,Q_R)}{Q_PQ_R} & for \quad x_1\le x\le x_2\\[2ex] \dfrac{x_{Max}-x}{Q_PQ_R} & for \quad x_2<x\le x_{Max}\\[2ex] 0 & otherwise;\end{cases} \tag{4.2}$$

and

$$\begin{aligned} x_1 &= Min(Q_P,Q_R);\\ x_{Max} &= Q_P+Q_R;\\ x_2 &= x_{Max}-x_1. \end{aligned} \tag{4.3}$$

It is easy to see that $f(.)\ge 0$ for all x. It is also true that

$$\int_0^{x_{Max}} f(I(t)=SW-x)dx=1.$$

To show this, we begin by integrating the first term of f(.) with respect to x.

$$\int_0^{x_1}\frac{x}{Q_PQ_R}dx=\frac{1}{Q_PQ_R}\frac{x^2}{2}\bigg|_0^{x_1}=\frac{x_1^2}{2Q_PQ_R}=\frac{(Min(Q_P,Q_R))^2}{2Q_PQ_R}. \tag{4.4}$$

Integrating the second term with respect to x gives

$$\begin{aligned}\int_{x_1}^{x_2}\frac{Min(Q_P,Q_R)}{Q_PQ_R}dx &= \frac{Min(Q_P,Q_R)}{Q_PQ_R}x\Big|_{x_1}^{x_2}=\frac{Min(Q_P,Q_R)}{Q_PQ_R}(x_2-x_1)\\ &=\frac{Min(Q_P,Q_R)}{Q_PQ_R}(x_{Max}-2x_1)\\ &=\frac{Min(Q_P,Q_R)}{Q_PQ_R}[Q_P+Q_R-Min(Q_P,Q_R)].\end{aligned} \tag{4.5}$$

Integrating the third term gives

$$\int_{x_2}^{x_{Max}} \frac{x_{Max} - x}{Q_P Q_R} dx = \frac{x_{Max} x}{Q_P Q_R}\Bigg|_{x_2}^{x_{Max}} - \frac{x^2}{2Q_P Q_R}\Bigg|_{x_2}^{x_{Max}}$$

$$= \frac{x_{Max}^2}{Q_P Q_R} - \frac{x_{Max} x_2}{Q_P Q_R} - \frac{x_{Max}^2}{2Q_P Q_R} + \frac{x_2^2}{2Q_P Q_R} = \frac{x_{Max}^2}{2Q_P Q_R} - \frac{x_{Max} x_1}{Q_P Q_R} + \frac{x_2^2}{2Q_P Q_R} \tag{4.6}$$

$$= \frac{(x_{Max} - x_2)^2}{2Q_P Q_R} = \frac{x_1^2}{2Q_P Q_R} = \frac{(Min(Q_P, Q_R))^2}{2Q_P Q_R}.$$

Summing the results of equations (4.4), (4.5), and (4.6) gives

$$\frac{(Min(Q_P, Q_R))^2}{2Q_P Q_R} + \frac{Min(Q_P, Q_R)}{Q_P Q_R}[Q_P + Q_R - 2Min(Q_P, Q_R)] + \frac{(Min(Q_P, Q_R))^2}{2Q_P Q_R}$$

$$= \frac{Min(Q_P, Q_R)}{Q_P Q_R}[Q_P + Q_R - Min(Q_P, Q_R)].$$

If $Min(Q_P, Q_R) = Q_P$ then

$$\frac{Min(Q_P, Q_R)}{Q_P Q_R}[Q_P + Q_R - Min(Q_P, Q_R)] = \frac{Q_P}{Q_P Q_R}[Q_P + Q_R - Q_P] = \frac{Q_P Q_R}{Q_P Q_R} = 1.$$

If $Min(Q_P, Q_R) = Q_R$ then

$$\frac{Min(Q_P, Q_R)}{Q_P Q_R}[Q_P + Q_R - Min(Q_P, Q_R)] = \frac{Q_R}{Q_P Q_R}[Q_P + Q_R - Q_R] = \frac{Q_R Q_P}{Q_P Q_R} = 1.$$

Therefore, it is true that

$$\int_0^{x_{Max}} f(I(t) = SW - x) dx = 1.$$

As a consequence, $f(I(t) = SW - x)$ satisfies all of the requirement to be a density function.

C. The Net Inventory Distribution

Baker's equation (38) (page 30, Reference.[1]) provides the structure for this approximation to that Poisson case. We will assume that the demand during the aggregate lead time is Normally distributed with a mean of μ and a standard deviation of σ. Now if this is to be an approximation to the Poisson case then $\sigma = \sqrt{\mu}$ should be true. However, there are items managed by the Navy's Inventory Control Point which don't meet that condition. Thus, we will assume that σ can take on any value but that μ will be sufficiently large that it will exceed 3σ so that there will be a negligible probability of a negative demand during the aggregate lead time.

Finally, we will follow the approach of Hadley and Whitin [3] in assuming that demand is truly continuous (see Chapter 4 page 192) rather than attempting to deal with a continuity correction.

We first rewrite equation (2.1) which is Baker's equation (38) as follows:

$$f[N(t)=SW-z]=\begin{cases} \dfrac{1}{Q_PQ_R}\displaystyle\int_{j=0}^{z} jf(z-j)dj & \text{for } 0\le z\le x_1 \\[2ex] \dfrac{1}{Q_PQ_R}\displaystyle\int_{0}^{x_1} jf(z-j)dj \\ +\dfrac{\min(Q_P,Q_R)}{Q_PQ_R}\displaystyle\int_{x_1}^{z} f(z-j)dj & \text{for } x_1\le z\le x_2 \\[2ex] \dfrac{1}{Q_PQ_R}\displaystyle\int_{0}^{x_1} jf(z-j)dj \\ +\dfrac{\min(Q_P,Q_R)}{Q_PQ_R}\displaystyle\int_{x_1}^{x_2} f(z-j)dj \\ +\dfrac{1}{Q_PQ_R}\displaystyle\int_{x_2}^{z} (x_{Max}-j)f(z-j)dj & \text{for } z\ge x_2 \end{cases} \quad . \tag{4.7}$$

We begin with the first part of this equation, the case where $0 \le z \le x_1$. Note that, as in Chapter 2, z represents the amount that the net inventory is below SW at any given time t. In addition, under the current assumption that demand during the aggregate lead time is Normal, it follows that if x is the demand then

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}; \tag{4.8}$$

where μ is the mean demand during the aggregate lead time and σ is the standard deviation. The first line of equation (4.7) when equation (4.8) is introduced is

$$\frac{1}{Q_PQ_R}\int_0^z jf(z-j)dj=\frac{1}{\sigma\sqrt{2\pi}Q_PQ_R}\int_0^z je^{-\frac{1}{2}\left(\frac{z-j-\mu}{\sigma}\right)^2}dj. \tag{4.9}$$

The right-hand side of equation (4.9) tends to be awkward to use, especially in developing the integral upper and lower bounds when converting (4.8) to a standard Normal (i.e., N(0,1)) density function. Therefore, let $v = z - j$. Then $j = z - v$ and $dj = -dv$. When $j = 0, v = z$ and when $j = z, v = 0$. The left-hand side of equation (4.9) changes therefore to

$$\frac{1}{Q_P Q_R}\int_0^z jf(z-j)dj = \frac{1}{Q_P Q_R}\int_0^z (z-v)f(v)dv. \tag{4.10}$$

Next define

$$r = \frac{v-\mu}{\sigma} \quad and \quad \phi(r) = \frac{1}{\sqrt{2\pi}}e^{-\frac{r^2}{2}}.$$

Then,

$$f(v) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{v-\mu}{\sqrt{\mu}}\right)^2} = \frac{1}{\sigma}\phi(r). \tag{4.11}$$

From the definition of r, it follows that $dr = \frac{dv}{\sigma}$ and therefore f(v)dv = ϕ(r)dr since the σ terms cancel.

Next, we need the upper and lower bounds for r in the integral. They are obtained from substituting the bounds for v into the equation defining r above. When $v = 0 \; and \; v = z$,

$$r = \frac{-\mu}{\sigma} \qquad and \qquad r = \frac{z-\mu}{\sigma}, \text{ respectively}$$

Finally, we need to transform $z - v$ into a function of r. From the definition of r it follows that

$$v = \mu + r\sigma \quad and\ therefore \quad z - v = z - \mu - r\sigma.$$

Putting all of this information together,

$$\frac{1}{Q_P Q_R}\int_0^z (z-v)f(v)dv = \frac{1}{Q_P Q_R}\int_{\frac{-\mu}{\sigma}}^{\frac{z-\mu}{\sigma}} (z-\mu-r\sigma)\phi(r)dr. \tag{4.12}$$

The first part of the right-hand side of equation (4.12) is:

$$\frac{1}{Q_P Q_R}\int_{\frac{-\mu}{\sigma}}^{\frac{z-\mu}{\sigma}} (z-\mu)\phi(r)dr = \frac{(z-\mu)}{Q_P Q_R}\int_{\frac{-\mu}{\sigma}}^{\frac{z-\mu}{\sigma}} \phi(r)dr = \frac{(z-\mu)}{Q_P Q_R}\left[\Phi\left(\frac{-\mu}{\sigma}\right)-\Phi\left(\frac{z-\mu}{\sigma}\right)\right]; \tag{4.13}$$

where $\Phi(u) \equiv \int_u^\infty \varphi(r)dr$. The second part of the right-hand side makes use of

Identity 1 of Appendix 4 of Reference [3].

$$\frac{-1}{Q_P Q_R}\int_{\frac{-\mu}{\sigma}}^{\frac{z-\mu}{\sigma}} r\sigma\,\varphi(r)dr = \frac{-\sigma}{Q_P Q_R}\int_{\frac{-\mu}{\sigma}}^{\frac{z-\mu}{\sigma}} r\phi(r)dr = \frac{-\sigma}{Q_P Q_R}\left[\phi\left(\frac{-\mu}{\sigma}\right)-\phi\left(\frac{z-\mu}{\sigma}\right)\right]. \tag{4.14}$$

Adding equations (4.13) and (4.14) gives the final form of equation (4.12).

$$\begin{aligned}\frac{1}{Q_P Q_R}\int_0^z (z-v)f(v)dv &= \frac{1}{Q_P Q_R}\int_{\frac{-\mu}{\sigma}}^{\frac{z-\mu}{\sigma}} (z-\mu-r\sigma)\phi(r)dr \\ &= \frac{z-\mu}{Q_p Q_R}\left[\Phi\left(\frac{-\mu}{\sigma}\right)-\Phi\left(\frac{z-\mu}{\sigma}\right)\right]-\frac{\sigma}{Q_P Q_R}\left[\phi\left(\frac{-\mu}{\sigma}\right)-\phi\left(\frac{z-\mu}{\sigma}\right)\right].\end{aligned} \tag{4.15}$$

For the case when $x_1 \le z \le x_2$, the density function for the net inventory is (see equation(4.7))

$$\frac{1}{Q_P Q_R}\int_0^{x_1} jf(z-j)dj + \frac{x_1}{Q_P Q_R}\int_{x_1}^z f(z-j)dj. \tag{4.16}$$

Following the same arguments as for the derivation leading to (4.15), the final form for the first part is

$$\frac{1}{Q_P Q_R}\int_0^{x_1} jf(z-j)dj = \frac{z-\mu}{Q_P Q_R}\left[\Phi\left(\frac{z-x_1-\mu}{\sigma}\right)-\Phi\left(\frac{z-\mu}{\sigma}\right)\right] - \frac{\sigma}{Q_P Q_R}\left[\phi\left(\frac{z-x_1-\mu}{\sigma}\right)-\phi\left(\frac{z-\mu}{\sigma}\right)\right]. \tag{4.17}$$

The final form for the second part is

$$\frac{x_1}{Q_P Q_R}\int_{x_1}^{z} f(z-j)dj = \frac{x_1}{Q_P Q_R}\left[\Phi\left(\frac{-\mu}{\sqrt{\mu}}\right)-\Phi\left(\frac{z-x_1-\mu}{\sqrt{\mu}}\right)\right]. \tag{4.18}$$

Adding equations (4.17) and (4.18) give the final form for equation (4.16).

$$\begin{aligned}\frac{1}{Q_P Q_R}\int_0^{x_1} jf(z-j)dj + \frac{x_1}{Q_P Q_R}\int_{x_1}^{z} f(z-j)dj = {} & \frac{z-\mu}{Q_P Q_R}\left[\Phi\left(\frac{z-x_1-\mu}{\sigma}\right)-\Phi\left(\frac{z-\mu}{\sigma}\right)\right] \\ & - \frac{\sigma}{Q_P Q_R}\left[\phi\left(\frac{z-x_1-\mu}{\sigma}\right)-\phi\left(\frac{z-\mu}{\sigma}\right)\right] \\ & + \frac{x_1}{Q_P Q_R}\left[\Phi\left(\frac{-\mu}{\sigma}\right)-\Phi\left(\frac{z-x_1-\mu}{\sigma}\right)\right].\end{aligned} \tag{4.19}$$

For the case where $x_2 \le z$, the last part of equation (4.7) applies. It is:

$$\begin{aligned}&\frac{1}{Q_P Q_R}\int_0^{x_1} jf(z-j)dj + \frac{x_1}{Q_P Q_R}\int_{x_1}^{x_2} f(z-j)dj \\ &+ \frac{1}{Q_P Q_R}\int_{x_2}^{Min(z,x_{Max})} (x_{Max}-j)f(z-j)dj.\end{aligned} \tag{4.20}$$

In this case, the first term is a repeat of the first part of equation (4.16). The second term is similar to the last part of equation (4.19) but its upper bound changes now to x_2. Therefore, their sum in final form is

$$\begin{aligned}&\frac{z-\mu}{Q_P Q_R}\left[\Phi\left(\frac{z-x_1-\mu}{\sigma}\right)-\Phi\left(\frac{z-\mu}{\sigma}\right)\right] \\ &- \frac{\sigma}{Q_P Q_R}\left[\phi\left(\frac{z-x_1-\mu}{\sigma}\right)-\phi\left(\frac{z-\mu}{\sigma}\right)\right] \\ &+ \frac{x_1}{Q_P Q_R}\left[\Phi\left(\frac{z-x_2-\mu}{\sigma}\right)-\Phi\left(\frac{z-x_1-\mu}{\sigma}\right)\right].\end{aligned} \tag{4.21}$$

The remaining term of (4.20), upon conversion of j to $z-v$, is

$$\frac{x_{Max}-z}{Q_P Q_R}\int_{z-Min(z,x_{Max})}^{z-x_2} f(v)dv+\frac{1}{Q_P Q_R}\int_{z-Min(z,x_{Max})}^{z-x_2} vf(v)dv,$$

which, when $v=\mu+r\sigma$, becomes

$$\frac{x_{Max}-z+\mu}{Q_P Q_R}\int_{\frac{z-Min(z,x_{Max})-\mu}{\sigma}}^{\frac{z-x_2-\mu}{\sigma}} \phi(r)dr+\frac{\sigma}{Q_P Q_R}\int_{\frac{z-Min(z,x_{Max})-\mu}{\sigma}}^{\frac{z-x_2-\mu}{\sigma}} r\phi(r)dr.$$

This then reduces to the following final form:

$$\begin{aligned}&\frac{x_{Max}-z+\mu}{Q_P Q_R}\left[\Phi\left(\frac{z-Min(z,x_{Max})-\mu}{\sigma}\right)-\Phi\left(\frac{z-x_2-\mu}{\sigma}\right)\right]\\&+\frac{\sigma}{Q_P Q_R}\left[\phi\left(\frac{z-Min(z,x_{Max})-\mu}{\sigma}\right)-\phi\left(\frac{z-x_2-\mu}{\sigma}\right)\right].\end{aligned} \tag{4.22}$$

Summing equations (4.21) and (4.22) provides the final form for equation (4.20).

$$\begin{aligned}&\frac{1}{Q_P Q_R}\int_0^{x_1} jf(z-j)dj+\frac{x_1}{Q_P Q_R}\int_{x_1}^{x_2} f(z-j)dj\\&+\frac{1}{Q_P Q_R}\int_{x_2}^{Min(z,x_{Max})}(x_{Max}-j)f(z-j)dj\\&=\frac{z-\mu}{Q_P Q_R}\left[\Phi\left(\frac{z-x_1-\mu}{\sigma}\right)-\Phi\left(\frac{z-\mu}{\sigma}\right)\right]\\&-\frac{\sigma}{Q_P Q_R}\left[\phi\left(\frac{z-x_1-\mu}{\sigma}\right)-\phi\left(\frac{z-\mu}{\sigma}\right)\right]\\&+\frac{x_1}{Q_P Q_R}\left[\Phi\left(\frac{z-x_2-\mu}{\sigma}\right)-\Phi\left(\frac{z-x_1-\mu}{\sigma}\right)\right]\\&+\frac{x_{Max}-z+\mu}{Q_P Q_R}\left[\Phi\left(\frac{z-Min(z,x_{Max})-\mu}{\sigma}\right)-\Phi\left(\frac{z-x_2-\mu}{\sigma}\right)\right]\\&+\frac{\sigma}{Q_P Q_R}\left[\phi\left(\frac{z-Min(z,x_{Max})-\mu}{\sigma}\right)-\phi\left(\frac{z-x_2-\mu}{\sigma}\right)\right].\end{aligned} \tag{4.23}$$

The final components of the density function for the net inventory are equations (4.15), (4.19), and (4.23). However, to reduce the visual complexity of the density function for the net inventory we will use a shorthand notation; namely,

$$\hat{x} \equiv \frac{x-\mu}{\sigma}. \tag{4.24}$$

With this notation the density function for the net inventory assuming a Normal distribution for the demand during the aggregate lead time can be written as:

$$f[N(t)=SW-z]=\begin{cases} \dfrac{z-\mu}{Q_PQ_R}\left[\Phi(\hat{0})-\Phi(\hat{z})\right]-\dfrac{\sigma}{Q_PQ_R}\left[\phi(\hat{0})-\phi(\hat{z})\right] & \textit{for } 0\le z\le x_1; \\ \\ \dfrac{z-\mu}{Q_PQ_R}\left[\Phi(\widehat{z-x_1})-\Phi(\hat{z})\right]-\dfrac{\sigma}{Q_PQ_R}\left[\phi(\widehat{z-x_1})-\phi(\hat{z})\right] & \\ +\dfrac{x_1}{Q_PQ_R}\left[\Phi(\hat{0})-\Phi(\widehat{z-x_1})\right] & \textit{for } x_1\le z\le x_2; \\ \\ \dfrac{z-\mu}{Q_PQ_R}\left[\Phi(\widehat{z-x_1})-\Phi(\hat{z})\right]-\dfrac{\sigma}{Q_PQ_R}\left[\phi(\widehat{z-x_1})-\phi(\hat{z})\right] & \\ +\dfrac{x_1}{Q_PQ_R}\left[\Phi(\widehat{z-x_2})-\Phi(\widehat{z-x_1})\right] & \\ +\dfrac{x_{Max}-z+\mu}{Q_PQ_R}\left[\Phi(\widehat{z-Min(z,x_{Max})})-\Phi(\widehat{z-x_2})\right] & \\ +\dfrac{\sigma}{Q_PQ_R}\left[\phi(\widehat{z-Min(z,x_{Max})})-\phi(\widehat{z-x_2})\right] & \textit{for } x_2\le z. \end{cases} \tag{4.25}$$

Finally, as we did with equation (2.3), we will rewrite equation (4.25) with the Q_PQ_R term removed to simplify the derivation of $P_{OUT}(SW)$.

$$
Q_P Q_R f[N(t)=SW-z]=\begin{cases}
(z-\mu)\left[\Phi(\hat{0})-\Phi(\hat{z})\right]-\sigma\left[\phi(\hat{0})-\phi(\hat{z})\right] & \text{for } 0\le z\le x_1; \\[2ex]
(z-\mu)\left[\Phi(\widehat{z-x_1})-\Phi(\hat{z})\right]-\sigma\left[\phi(\widehat{z-x_1})-\phi(\hat{z})\right] & \\
+x_1\left[\Phi(\hat{0})-\Phi(\widehat{z-x_1})\right] & \text{for } x_1\le z\le x_2; \\[2ex]
(z-\mu)\left[\Phi(\widehat{z-x_1})-\Phi(\hat{z})\right]-\sigma\left[\phi(\widehat{z-x_1})-\phi(\hat{z})\right] & \\
+x_1\left[\Phi(\widehat{z-x_2})-\Phi(\widehat{z-x_1})\right] & \\
+(x_{Max}-z+\mu)\left[\Phi(\widehat{z-Min(z,x_{Max})})-\Phi(\widehat{z-x_2})\right] & \\
+\sigma\left[\phi(\widehat{z-Min(z,x_{Max})})-\phi(\widehat{z-x_2})\right] & \text{for } x_2\le z.
\end{cases}
\tag{4.26}
$$

D. Derivation of $P_{OUT}(SW)$

1. The case of $0\le SW\le x_1$

We begin by writing $P_{OUT}(SW)$ in the structure of equation (2.12).

$$
\begin{aligned}
P_{OUT}(SW)=&\frac{1}{Q_PQ_R}\int_{SW}^{x_1}\left\{(z-\mu)\left[\Phi(\hat{0})-\Phi(\hat{z})\right]-\sigma\left[\phi(\hat{0})-\phi(\hat{z})\right]\right\}dz \\
&+\frac{1}{Q_PQ_R}\int_{x_1}^{x_2}\left\{\begin{array}{l}(z-\mu)\left[\Phi(\widehat{z-x_1})-\Phi(\hat{z})\right]-\sigma\left[\phi(\widehat{z-x_1})-\phi(\hat{z})\right] \\ +x_1\left[\Phi(\hat{0})-\Phi(\widehat{z-x_1})\right]\end{array}\right\}dz \\
&+\frac{1}{Q_PQ_R}\int_{x_2}^{\infty}\left\{\begin{array}{l}(z-\mu)\left[\Phi(\widehat{z-x_1})-\Phi(\hat{z})\right]-\sigma\left[\phi(\widehat{z-x_1})-\phi(\hat{z})\right] \\ +x_1\left[\Phi(\widehat{z-x_2})-\Phi(\widehat{z-x_1})\right] \\ +(x_{Max}-z+\mu)\left[\Phi(\widehat{z-Min(z,x_{Max})})-\Phi(\widehat{z-x_2})\right] \\ +\sigma\left[\phi(\widehat{z-Min(z,x_{Max})})-\phi(\widehat{z-x_2})\right]\end{array}\right\}dz.
\end{aligned}
\tag{4.27}
$$

As we did in Chapter 2 to get equation (2.13), we collect common terms and rewrite equation (4.27) as:

$$
\begin{aligned}
Q_P Q_R P_{OUT}(SW) = &\int_{SW}^{x_1} (z-\mu)\Phi(\hat{0})dz - \int_{SW}^{\infty} (z-\mu)\Phi(\hat{z})dz - \sigma \int_{SW}^{x_1} \phi(\hat{0})dz \\
&+ \sigma \int_{SW}^{\infty} \phi(\hat{z})dz + \int_{x_1}^{\infty} (z-\mu)\Phi(\widehat{z-x_1})dz - \sigma \int_{x_1}^{\infty} \phi(\widehat{z-x_1})dz \\
&+ x_1 \int_{x_1}^{x_2} \Phi(\hat{0})dz - x_1 \int_{x_1}^{\infty} \Phi(\widehat{z-x_1})dz + x_1 \int_{x_2}^{\infty} \Phi(\widehat{z-x_2})dz \\
&+ \int_{x_2}^{x_{Max}} (x_{Max} - z + \mu)\Phi(\hat{0})dz + \sigma \int_{x_2}^{x_{Max}} \phi(\hat{0})dz \\
&- \int_{x_2}^{\infty} (x_{Max} - z + \mu)\Phi(\widehat{z-x_2})dz - \sigma \int_{x_2}^{\infty} \phi(\widehat{z-x_2})dz \\
&+ \int_{x_{Max}}^{\infty} (x_{Max} - z + \mu)\Phi(\widehat{z-x_{Max}})dz + \sigma \int_{x_{Max}}^{\infty} \phi(\widehat{z-x_{Max}})dz .
\end{aligned}
\tag{4.28}
$$

We will consider the terms in equation (4.28) which contain either $\Phi(\hat{0})$ and $\phi(\hat{0})$ as constants. We will evaluate them first.

$$
\int_{SW}^{x_1} (z-\mu)\Phi(\hat{0})dz = \Phi(\hat{0})\left\{\frac{z^2}{2} - \mu z\right\}\Bigg|_{SW}^{x_1} = \Phi(\hat{0})\left\{\frac{x_1^2 - SW^2}{2} - \mu(x_1 - SW)\right\};
$$

$$
-\sigma \int_{SW}^{x_1} \phi(\hat{0})dz = -\sigma\phi(\hat{0})[x_1 - SW];
$$

$$
x_1 \int_{x_1}^{x_2} \Phi(\hat{0})dz = x_1\Phi(\hat{0})[x_2 - x_1] = \Phi(\hat{0})\left[x_1 x_2 - x_1^2\right].
$$

In the next term we make use of the definition of x_2 given by equation (4.3). We rewrite it as

$$
x_1 = x_{Max} - x_2 .
$$

$$\int_{x_2}^{x_{Max}} (x_{Max} - z + \mu)\Phi(\hat{0})dz = (x_{Max} + \mu)\Phi(\hat{0})[x_{Max} - x_2] - \Phi(\hat{0})\left[\frac{x_{Max}^2}{2} - \frac{x_2^2}{2}\right]$$

$$= \Phi(\hat{0})(x_{Max} + \mu)x_1 - \Phi(\hat{0})\frac{(x_{Max} - x_2)(x_{Max} + x_2)}{2}$$

$$= \Phi(\hat{0})\left[x_{Max}x_1 + \mu x_1 - \frac{x_{Max}x_1}{2} - \frac{x_1 x_2}{2}\right]$$

$$= \Phi(\hat{0})\left[\mu x_1 + \frac{x_{Max}x_1}{2} - \frac{x_1 x_2}{2}\right] = \Phi(\hat{0})\left[\mu x_1 + \frac{(x_{Max} - x_2)x_1}{2}\right]$$

$$= \Phi(\hat{0})\left[\mu x_1 + \frac{x_1^2}{2}\right];$$

$$\sigma \int_{x_2}^{x_{Max}} \phi(\hat{0})dz = \sigma\phi(\hat{0})[x_{Max} - x_2] = \sigma\phi(\hat{0})x_1 .$$

Summing all of these "constants" gives

$$\Phi(\hat{0})\left\{\frac{x_1^2 - SW^2}{2} - \mu(x_1 - SW)\right\} - \sigma\phi(\hat{0})[x_1 - SW]$$

$$+ \Phi(\hat{0})\left[x_1 x_2 - x_1^2\right] + \Phi(\hat{0})\left[\mu x_1 + \frac{x_1^2}{2}\right] + \sigma\phi(\hat{0})x_1 \qquad (4.29)$$

$$= \Phi(\hat{0})\left\{-\frac{SW^2}{2} + \mu SW + x_1 x_2\right\} + \sigma\phi(\hat{0})SW .$$

Next, recalling equation(4.24), we know that

$$\hat{z} = \frac{z - \mu}{\sigma},$$

and, therefore,

$$d\hat{z} = \frac{dz}{\sigma}, \quad and \quad \widehat{SW} = \frac{SW - \mu}{\sigma} .$$

As a consequence, we write the second term of equation (4.28) as follows:

$$-\int_{SW}^{\infty} (z - \mu)\Phi(\hat{z})dz = -\sigma \int_{SW}^{\infty} \left(\frac{z - \mu}{\sigma}\right)\Phi(\hat{z})dz = -\sigma^2 \int_{\widehat{SW}}^{\infty} \hat{z}\Phi(\hat{z})d\hat{z} = -\sigma^2\gamma(\widehat{SW}) ; \qquad (4.30)$$

where, using Identity 7 from Appendix 4 of Reference [3], we define

$$\gamma(\hat{u}) \equiv \int_{\hat{u}}^{\infty} \hat{r}\Phi(\hat{r})d\hat{r} \equiv \frac{1}{2}\left\{(1-\hat{u}^2)\Phi(\hat{u}) + \hat{u}\phi(\hat{u})\right\}.$$

The fourth term of equation (4.28) can be easily reduced.

$$\sigma \int_{SW}^{\infty} \phi(\hat{z})dz = \sigma^2 \int_{\widehat{SW}}^{\infty} \phi(\hat{z})d\hat{z} = \sigma^2\Phi(\widehat{SW}). \tag{4.31}$$

To reduce $\int_{x_1}^{\infty}(z-\mu)\Phi(\widehat{z-x_1})dz$ we need to introduce a change of variable.

Let $u = z - x_1$. Then $\hat{u} = \dfrac{z - x_1 - \mu}{\sigma}$, and therefore $z - \mu = x_1 + \sigma\hat{u}$. Finally, when $z = x_1$, $\hat{u} = \hat{0}$. We also need to add a definition for $\alpha(\hat{u})$. It is

$$\alpha(\hat{u}) \equiv \int_{\hat{u}}^{\infty} \Phi(\hat{r})d\hat{r} = \phi(\hat{u}) - \hat{u}\Phi(\hat{u}). \tag{4.32}$$

Therefore, we can write

$$\begin{aligned}\int_{x_1}^{\infty}(z-\mu)\Phi(\widehat{z-x_1})dz &= \sigma\int_{\hat{0}}^{\infty}(x_1 + \sigma\hat{u})\Phi(\hat{u})d\hat{u} \\ &= \sigma x_1\int_{\hat{0}}^{\infty}\Phi(\hat{u})d\hat{u} + \sigma^2\int_{\hat{0}}^{\infty}\hat{u}\Phi(\hat{u})d\hat{u} \\ &= \sigma x_1\alpha(\hat{0}) + \sigma^2\gamma(\hat{0}).\end{aligned} \tag{4.33}$$

Next, we need to reduce $-\sigma\int_{x_1}^{\infty}\phi(\widehat{z-x_1})dz$. We get

$$-\sigma\int_{x_1}^{\infty}\phi(\widehat{z-x_1})dz = -\sigma^2\int_{\hat{0}}^{\infty}\phi(\hat{u})d\hat{u} = -\sigma^2\Phi(\hat{0}). \tag{4.34}$$

The eighth and ninth terms from equation (4.28) cancel as shown below. This is a

consequence of letting $\hat{u} = \dfrac{z - x_1 - \mu}{\sigma}$ in the eighth term and then $\hat{u} = \dfrac{z - x_2 - \mu}{\sigma}$ in the ninth term.

$$-x_1 \int_{x_1}^{\infty} \Phi(\widehat{z - x_1})dz + x_1 \int_{x_2}^{\infty} \Phi(\widehat{z - x_2})dz = -x_1\sigma \int_{\hat{0}}^{\infty} \Phi(\hat{u})d\hat{u} + x_1\sigma \int_{\hat{0}}^{\infty} \Phi(\hat{u})d\hat{u} = 0.$$

The last two lines of terms in equation (4.28) simplify down to one term because the rest of the reduced terms cancel. As in the preceding derivation, $\hat{u}$ is defined differently for each part. First,

$$\begin{aligned}
-\int_{x_2}^{\infty} (x_{Max} - z + \mu)\Phi(\widehat{z - x_2})dz &= -\sigma x_{Max} \int_{\hat{0}}^{\infty} \Phi(\hat{u})d\hat{u} + \sigma x_2 \int_{\hat{0}}^{\infty} \Phi(\hat{u})d\hat{u} + \sigma^2 \int_{\hat{0}}^{\infty} \hat{u}\Phi(\hat{u})d\hat{u} \\
&= -\sigma x_{Max}\alpha(\hat{0}) + \sigma x_2\alpha(\hat{0}) + \sigma^2\gamma(\hat{0}) \\
&= -\sigma x_1\alpha(\hat{0}) + \sigma^2\gamma(\hat{0}).
\end{aligned}$$

$$\begin{aligned}
\int_{x_{Max}}^{\infty} (x_{Max} - z + \mu)\Phi(\widehat{z - x_{Max}})dz &= \sigma x_{Max} \int_{\hat{0}}^{\infty} \Phi(\hat{u})d\hat{u} - \sigma x_{Max} \int_{\hat{0}}^{\infty} \Phi(\hat{u})d\hat{u} - \sigma^2 \int_{\hat{0}}^{\infty} \hat{u}\Phi(\hat{u})d\hat{u} \\
&= \sigma x_{Max}\alpha(\hat{0}) - \sigma x_{Max}\alpha(\hat{0}) - \sigma^2\gamma(\hat{0}) \\
&= -\sigma^2\gamma(\hat{0}).
\end{aligned}$$

Finally,

$$-\sigma \int_{x_2}^{\infty} \phi(\widehat{z - x_2})dz + \sigma \int_{x_{Max}}^{\infty} \phi(\widehat{z - x_{Max}})dz = -\sigma^2 \int_{\hat{0}}^{\infty} \phi(\hat{u})d\hat{u} + \sigma^2 \int_{\hat{0}}^{\infty} \phi(\hat{u})d\hat{u} = 0.$$

When the terms in the last two lines of equation (4.28) are summed the result is merely $-\sigma x_1\alpha(\hat{0})$.

$$-(x_{Max}-z+\mu)\int_{x_2}^{\infty}\Phi(\widehat{z-x_2})dz-\sigma\int_{x_2}^{\infty}\phi(\widehat{z-x_2})dz$$

$$+(x_{Max}-z+\mu)\int_{x_{Max}}^{\infty}\Phi(\widehat{z-x_{Max}})dz+\sigma\int_{x_{Max}}^{\infty}\phi(\widehat{z-x_{Max}})dz \qquad (4.35)$$

$$=-\sigma x_1\alpha(\hat{0})+\sigma^2\gamma(\hat{0})-\sigma^2\gamma(\hat{0})=-\sigma x_1\alpha(\hat{0}).$$

Summing the results of equations(4.29), (4.30), (4.31), (4.33), (4.34), and (4.35) gives the final form for $P_{OUT}(SW)$ for the case of $0\le SW\le x_1$.

$$\begin{aligned}
P_{OUT}(SW)=\Phi(\hat{0})&\left\{-\frac{SW^2}{2}+\mu SW+x_1x_2\right\}+\sigma SW\phi(\hat{0})\\
&-\sigma^2\gamma(\widehat{SW})+\sigma^2\Phi(\widehat{SW})+\sigma x_1\alpha(\hat{0})\\
&+\sigma^2\gamma(\hat{0})-\sigma^2\Phi(\hat{0})-\sigma x_1\alpha(\hat{0})+\sigma^2\gamma(\hat{0})-\sigma^2\gamma(\hat{0}) \qquad (4.36)\\
&=\Phi(\hat{0})\left\{-\frac{SW^2}{2}+\mu SW+x_1x_2\right\}+\sigma SW\phi(\hat{0})\\
&\quad-\sigma^2\left[\gamma(\widehat{SW})-\Phi(\widehat{SW})\right]+\sigma^2\left[\gamma(\hat{0})-\Phi(\hat{0})\right].
\end{aligned}$$

2. The case of $x_1\le SW\le x_2$

In this case the first part of equation (4.27) disappears and the second part has SW as the lower bound on the integrals. The initial form of $P_{OUT}(SW)$ is now

$$\begin{aligned}
P_{OUT}(SW)=&\frac{1}{Q_PQ_R}\int_{SW}^{x_2}\left\{\begin{array}{l}(z-\mu)\left[\Phi(\widehat{z-x_1})-\Phi(\hat{z})\right]-\sigma\left[\phi(\widehat{z-x_1})-\phi(\hat{z})\right]\\+x_1\left[\Phi(\hat{0})-\Phi(\widehat{z-x_1})\right]\end{array}\right\}dz\\
&+\frac{1}{Q_PQ_R}\int_{x_2}^{\infty}\left\{\begin{array}{l}(z-\mu)\left[\Phi(\widehat{z-x_1})-\Phi(\hat{z})\right]-\sigma\left[\phi(\widehat{z-x_1})-\phi(\hat{z})\right]\\+x_1\left[\Phi(\widehat{z-x_2})-\Phi(\widehat{z-x_1})\right]\\+(x_{Max}-z+\mu)\left[\Phi(\widehat{z-Min(z,x_{Max})})-\Phi(\widehat{z-x_2})\right]\\+\sigma\left[\phi(\widehat{z-Min(z,x_{Max})})-\phi(\widehat{z-x_2})\right]\end{array}\right\}dz\,.(4.37)
\end{aligned}$$

Collecting like terms, we write the equivalent form to equation (4.28) for this case.

$$
\begin{aligned}
Q_P Q_R P_{OUT}(SW) = &-\int_{SW}^{\infty}(z-\mu)\Phi(\hat{z})dz+\sigma\int_{SW}^{\infty}\phi(\hat{z})dz \\
&+\int_{SW}^{\infty}(z-\mu)\Phi(\widehat{z-x_1})dz-\sigma\int_{SW}^{\infty}\phi(\widehat{z-x_1})dz \\
&+x_1\int_{SW}^{x_2}\Phi(\hat{0})dz-x_1\int_{SW}^{\infty}\Phi(\widehat{z-x_1})dz+x_1\int_{x_2}^{\infty}\Phi(\widehat{z-x_2})dz \\
&+(x_{Max}-z+\mu)\int_{x_2}^{x_{Max}}\Phi(\hat{0})dz+\sigma\int_{x_2}^{x_{Max}}\phi(\hat{0})dz \\
&-(x_{Max}-z+\mu)\int_{x_2}^{\infty}\Phi(\widehat{z-x_2})dz-\sigma\int_{x_2}^{\infty}\phi(\widehat{z-x_2})dz \\
&+(x_{Max}-z+\mu)\int_{x_{Max}}^{\infty}\Phi(\widehat{z-x_{Max}})dz+\sigma\int_{x_{Max}}^{\infty}\phi(\widehat{z-x_{Max}})dz.
\end{aligned}
\tag{4.38}
$$

The last three lines of equation (4.38) are identical to the last three lines of equation (4.28). In addition, the first line of equation (4.38) is the same as the second and fourth terms of equation (4.28). Therefore, we need only derive the reduced forms for the second and third lines of equation (4.38).

For the terms of the second line, let $u = z - x_1$. Then, when $z = SW$, $u = SW - x_1$. The rest follows from the definition of $\hat{u}$ above (just preceding the definition of $\alpha(\hat{u})$ given by equation (4.32)).

$$
\begin{aligned}
\int_{SW}^{\infty}(z-\mu)\Phi(\widehat{z-x_1})dz &= \sigma\int_{\widehat{SW-x_1}}^{\infty}(x_1+\sigma\hat{u})\Phi(\hat{u})d\hat{u} \\
&= \sigma x_1\alpha(\widehat{SW-x_1})+\sigma^2\gamma(\widehat{SW-x_1}).
\end{aligned}
\tag{4.39}
$$

$$-\sigma \int_{SW}^{\infty} \phi(\widehat{z-x_1})dz = -\sigma^2 \int_{\widehat{SW-x_1}}^{\infty} \phi(\hat{u})d\hat{u} = -\sigma^2\Phi(\widehat{SW-x_1}). \tag{4.40}$$

The first term of the third line is a constant term;

$$x_1 \int_{SW}^{x_2} \Phi(\hat{0})dz = x_1\Phi(\hat{0})(x_2 - SW). \tag{4.41}$$

The second term of the third line is

$$-x_1 \int_{SW}^{\infty} \Phi(\widehat{z-x_1})dz = -\sigma x_1 \int_{\widehat{SW-x_1}}^{\infty} \Phi(\hat{u})d\hat{u} = -\sigma x_1 \alpha(\widehat{SW-x_1}). \tag{4.42}$$

The third term is the same as for the previous case. There it was not completely reduced since it cancelled the second term. Here, it needs to be reduced since its lower bound is different from the second term.

$$x_1 \int_{x_2}^{\infty} \Phi(\widehat{z-x_2})dz = x_1\sigma \int_{\hat{0}}^{\infty} \Phi(\hat{u})d\hat{u} = x_1\sigma\alpha(\hat{0}). \tag{4.43}$$

Therefore, adding the results of equations (4.39) through (4.43) gives

$$\begin{aligned} &\sigma x_1\alpha(\widehat{SW-x_1}) + \sigma^2\gamma(\widehat{SW-x_1}) - \sigma^2\Phi(\widehat{SW-x_1}) \\ &\quad + x_1\Phi(\hat{0})(x_2 - SW) - \sigma x_1\alpha(\widehat{SW-x_1}) + x_1\sigma\alpha(\hat{0}) \\ &\quad = \sigma^2\gamma(\widehat{SW-x_1}) - \sigma^2\Phi(\widehat{SW-x_1}) \\ &\qquad + x_1\Phi(\hat{0})(x_2 - SW) + x_1\sigma\alpha(\hat{0}) \end{aligned} \tag{4.44}$$

Combining this result with the unchanged derivations for the previous case, we get

$$\begin{aligned} P_{OUT}(SW) = {} & x_1\Phi(\hat{0})\left\{x_2 + \frac{x_1}{2} + \mu - SW\right\} + \sigma x_1\phi(\hat{0}) \\ & - \sigma^2\left[\gamma(\widehat{SW}) - \Phi(\widehat{SW})\right] \\ & + \sigma^2\left[\gamma(\widehat{SW-x_1}) - \Phi(\widehat{SW-x_1})\right]. \end{aligned} \tag{4.45}$$

3. The case of $x_2 \le SW \le x_{Max}$

Equation (4.37) can be easily modified to get the starting form of $P_{OUT}(SW)$ for this case.

$$P_{OUT}(SW)=\frac{1}{Q_PQ_R}\int_{SW}^{\infty}\left\{\begin{array}{l}(z-\mu)\left[\Phi(\widehat{z-x_1})-\Phi(\hat{z})\right]-\sigma\left[\phi(\widehat{z-x_1})-\phi(\hat{z})\right]\\ +x_1\left[\Phi(\widehat{z-x_2})-\Phi(\widehat{z-x_1})\right]\\ +(x_{Max}-z+\mu)\left[\Phi(\widehat{z-Min(z,x_{Max})})-\Phi(\widehat{z-x_2})\right]\\ +\sigma\left[\phi(\widehat{z-Min(z,x_{Max})})-\phi(\widehat{z-x_2})\right]\end{array}\right\}dz\,. \quad (4.46)$$

Because of the $Min(z,x_{Max})$ term it is appropriate to consider the interval between x_2 and x_{Max} by itself. Equation (4.46) can be rewritten as

$$\begin{aligned}Q_PQ_RP_{OUT}(SW)=&-\int_{SW}^{\infty}(z-\mu)\Phi(\hat{z})dz+\sigma\int_{SW}^{\infty}\phi(\hat{z})dz\\ &+\int_{SW}^{\infty}(z-\mu)\Phi(\widehat{z-x_1})dz-\sigma\int_{SW}^{\infty}\phi(\widehat{z-x_1})dz\\ &-x_1\int_{SW}^{\infty}\Phi(\widehat{z-x_1})dz+x_1\int_{SW}^{\infty}\Phi(\widehat{z-x_2})dz\\ &+(x_{Max}-z+\mu)\int_{SW}^{x_{Max}}\Phi(\hat{0})dz+\sigma\int_{SW}^{x_{Max}}\phi(\hat{0})dz\\ &-(x_{Max}-z+\mu)\int_{SW}^{\infty}\Phi(\widehat{z-x_2})dz-\sigma\int_{SW}^{\infty}\phi(\widehat{z-x_2})dz\\ &+(x_{Max}-z+\mu)\int_{x_{Max}}^{\infty}\Phi(\widehat{z-x_{Max}})dz+\sigma\int_{x_{Max}}^{\infty}\phi(\widehat{z-x_{Max}})dz\,.\end{aligned} \quad (4.47)$$

The first, second, and last lines are the same as for the previous case. The first term of the third line does not change either. The reduction of the remaining terms follows.

$$x_1 \int_{SW}^{\infty} \Phi(\widehat{z-x_2})dz = \sigma x_1 \int_{\widehat{SW-x_2}}^{\infty} \Phi(\hat{u})d\hat{u} = \sigma x_1 \alpha(\widehat{SW-x_2}). \qquad (4.48)$$

$$\int_{SW}^{x_{Max}} (x_{Max} - z + \mu)\Phi(\hat{0})dz = (x_{Max} + \mu)\Phi(\hat{0})(x_{Max} - SW) - \Phi(\hat{0})\frac{z^2}{2}\Big|_{SW}^{x_{Max}}$$

$$= (x_{Max} + \mu)\Phi(\hat{0})(x_{Max} - SW) - \Phi(\hat{0})\left\{\frac{x_{Max}^2}{2} - \frac{SW^2}{2}\right\} \qquad (4.49)$$

$$= \Phi(\hat{0})\left[\frac{1}{2}(x_{Max} - SW)^2 + \mu(x_{Max} - SW)\right].$$

$$\sigma \int_{SW}^{x_{Max}} \phi(\hat{0})dz = \sigma\phi(\hat{0})(x_{Max} - SW). \qquad (4.50)$$

$$-\int_{SW}^{\infty} (x_{Max} - z + \mu)\Phi(\widehat{z-x_2})dz$$

$$= -x_{Max}\sigma \int_{\widehat{SW-x_2}}^{\infty} \Phi(\hat{u})d\hat{u} + \sigma x_2 \int_{\widehat{SW-x_2}}^{\infty} \Phi(\hat{u})d\hat{u} + \sigma^2 \int_{\widehat{SW-x_2}}^{\infty} \hat{u}\Phi(\hat{u})d\hat{u}$$

$$= -x_{Max}\sigma\alpha(\widehat{SW-x_2}) + \sigma x_2\alpha(\widehat{SW-x_2}) + \sigma^2\gamma(\widehat{SW-x_2}) \qquad (4.51)$$

$$= -x_1\sigma\alpha(\widehat{SW-x_2}) + \sigma^2\gamma(\widehat{SW-x_2}).$$

$$-\sigma \int_{SW}^{\infty} \phi(\widehat{z-x_2})dz = -\sigma^2 \int_{\widehat{SW-x_2}}^{\infty} \phi(\hat{u})d\hat{u} = -\sigma^2\Phi(\widehat{SW-x_2}). \qquad (4.52)$$

Summing up the results of equations (4.48) through(4.52),

$$\sigma x_1\alpha(\widehat{SW-x_2}) + \Phi(\hat{0})\left\{\frac{(x_{Max} - SW)^2}{2} + \mu(x_{Max} - SW)\right\}$$

$$+ \sigma\phi(\hat{0})(x_{Max} - SW) - x_1\sigma\alpha(\widehat{SW-x_2}) + \sigma^2\gamma(\widehat{SW-x_2}) - \sigma^2\Phi(\widehat{SW-x_2})$$

$$= \Phi(\hat{0})\left\{\frac{(x_{Max} - SW)^2}{2} + \mu(x_{Max} - SW)\right\} + \sigma\phi(\hat{0})(x_{Max} - SW) \qquad (4.53)$$

$$+ \sigma^2\left[\gamma(\widehat{SW-x_2}) - \Phi(\widehat{SW-x_2})\right].$$

Summing the other terms which were unchanged from the previous case, we get

$$
\begin{aligned}
&\sigma x_1\alpha(\widehat{SW-x_1})+\sigma^2\left[\gamma(\widehat{SW-x_1})-\Phi(\widehat{SW-x_1})\right] \\
&\quad -\sigma^2\left[\gamma(\widehat{SW})-\Phi(\widehat{SW})\right]-\sigma x_1\alpha(\widehat{SW-x_1})-\sigma^2\left[\gamma(\hat{0})-\Phi(\hat{0})\right] \\
&\quad =\sigma^2\left[\gamma(\widehat{SW-x_1})-\Phi(\widehat{SW-x_1})\right]-\sigma^2\left[\gamma(\widehat{SW})-\Phi(\widehat{SW})\right] \\
&\quad -\sigma^2\left[\gamma(\hat{0})-\Phi(\hat{0})\right].
\end{aligned}
\tag{4.54}
$$

Finally, adding equations (4.53) and (4.54) gives final form for $P_{OUT}(SW)$ for the case of $x_2 < SW \leq x_{Max}$.

$$
\begin{aligned}
P_{OUT}(SW)=\Phi(\hat{0})&\left\{\frac{(x_{Max}-SW)^2}{2}+\mu(x_{Max}-SW)\right\}+\sigma\phi(\hat{0})(x_{Max}-SW) \\
&-\sigma^2\left[\gamma(\widehat{SW})-\Phi(\widehat{SW})\right]+\sigma^2\left[\gamma(\widehat{SW-x_1})-\Phi(\widehat{SW-x_1})\right] \\
&+\sigma^2\left[\gamma(\widehat{SW-x_2})-\Phi(\widehat{SW-x_2})\right]-\sigma^2\left[\gamma(\hat{0})-\Phi(\hat{0})\right].
\end{aligned}
\tag{4.55}
$$

4. The case of $SW \geq x_{Max}$.

This case is similar to the preceding one in the fact that equation (4.46) also applies here. Now, however, $Min(z,x_{Max}) = x_{Max}$. Thus, equation (4.46) takes on the following form:

$$
P_{OUT}(SW)=\frac{1}{Q_P Q_R}\int_{SW}^{\infty}\left\{\begin{aligned}
&(z-\mu)\left[\Phi(\widehat{z-x_1})-\Phi(\hat{z})\right]-\sigma\left[\phi(\widehat{z-x_1})-\phi(\hat{z})\right] \\
&+x_1\left[\Phi(\widehat{z-x_2})-\Phi(\widehat{z-x_1})\right] \\
&+(x_{Max}-z+\mu)\left[\Phi(\widehat{z-x_{Max}})-\Phi(\widehat{z-x_2})\right] \\
&+\sigma\left[\phi(\widehat{z-x_{Max}})-\phi(\widehat{z-x_2})\right]
\end{aligned}\right\}dz. \tag{4.56}
$$

The fourth line of equation (4.47) is now eliminated. The reduced version of equation (4.47) is given by equation (4.57) below. Only the terms in the last line of equation (4.57) have not been analyzed somewhere above.

$$
\begin{aligned}
Q_P Q_R P_{OUT}(SW) = &-\int_{SW}^{\infty} (z-\mu)\Phi(\hat{z})dz + \sigma \int_{SW}^{\infty} \phi(\hat{z})dz \\
&+ \int_{SW}^{\infty} (z-\mu)\Phi(\widehat{z-x_1})dz - \sigma \int_{SW}^{\infty} \phi(\widehat{z-x_1})dz \\
&- x_1 \int_{SW}^{\infty} \Phi(\widehat{z-x_1})dz + x_1 \int_{SW}^{\infty} \Phi(\widehat{z-x_2})dz \\
&- \int_{SW}^{\infty} (x_{Max}-z+\mu)\Phi(\widehat{z-x_2})dz - \sigma \int_{SW}^{\infty} \phi(\widehat{z-x_2})dz \\
&+ \int_{SW}^{\infty} (x_{Max}-z+\mu)\Phi(\widehat{z-x_{Max}})dz + \sigma \int_{SW}^{\infty} \phi(\widehat{z-x_{Max}})dz .
\end{aligned}
\tag{4.57}
$$

The terms of the last line of equation (4.57) are

$$
\begin{aligned}
\int_{SW}^{\infty} (x_{Max}-z+\mu)\Phi(\widehat{z-x_{Max}})dz &= -\sigma \int_{SW}^{\infty} \left\{\frac{z-x_{Max}-\mu}{\sigma}\right\}\Phi(\widehat{z-x_{Max}})dz \\
= -\sigma^2 \int_{\widehat{SW-x_{Max}}}^{\infty} \hat{u}\Phi(\hat{u})d\hat{u} &= -\sigma^2\gamma(\widehat{SW-x_{Max}}),
\end{aligned}
\tag{4.58}
$$

and

$$
\sigma \int_{SW}^{\infty} \phi(\widehat{z-x_{Max}})dz = \sigma^2 \int_{\widehat{SW-x_{Max}}}^{\infty} \phi(\hat{u})d\hat{u} = \sigma^2\Phi(\widehat{SW-x_{Max}}). \tag{4.59}
$$

The sum of equations (4.58) and (4.59) is

$$
\begin{aligned}
&\int_{SW}^{\infty} (x_{Max}-z+\mu)\Phi(\widehat{z-x_{Max}})dz + \sigma \int_{SW}^{\infty} \phi(\widehat{z-x_{Max}})dz \\
&\qquad = -\sigma^2\left[\gamma(\widehat{SW-x_{Max}}) - \Phi(\widehat{SW-x_{Max}})\right].
\end{aligned}
\tag{4.60}
$$

The final form for $P_{OUT}(SW)$ for the case of $x_{Max} \le SW$ is, therefore,

$$
\begin{aligned}
P_{OUT}(SW) = &-\sigma^2\left[\gamma(\widehat{SW}) - \Phi(\widehat{SW})\right] \\
&+\sigma^2\left[\gamma(\widehat{SW-x_1}) - \Phi(\widehat{SW-x_1})\right] \\
&+\sigma^2\left[\gamma(\widehat{SW-x_2}) - \Phi(\widehat{SW-x_2})\right] \\
&-\sigma^2\left[\gamma(\widehat{SW-x_{Max}}) - \Phi(\widehat{SW-x_{Max}})\right].
\end{aligned}
\tag{4.61}
$$

E. Summary

This completes the derivations of the equations for $P_{OUT}(SW)$ under the assumption that demand during the aggregate lead time is Normally distributed. For $0 \le SW \le x_1$, equation (4.36) applies; for $x_1 \le SW \le x_2$, equation (4.45) applies; for $x_2 \le SW \le x_{Max}$, equation (4.55) applies; and, for $x_{Max} \le SW$, equation (4.61) applies.

CHAPTER 5 – DERIVATION OF $B(SW)$ FOR THE NORMAL MODIFICATION

A. Introduction

This chapter presents the derivations of the formulas for all of the cases for the expected number of backorders at any instant of time, $B(SW)$, for the Baker model where demand during aggregate lead time is assumed to be Normally distributed. The density function for the Normal modification to the Baker model was developed in Chapter 4 and is given by equation (4.25). The derivations in this chapter will also rely heavily on the derivations in Chapter 3 which gave the formulas for $B(SW)$ for the Baker model [1]. In place of equation (3.1) we now have

$$B(SW) = \int_{SW}^{\infty} (z - SW) f[N(t) = SW - z] dz \tag{5.1}$$

Next, we can separate this equation into the following two parts;

$$B(SW) = \int_{SW}^{\infty} z f[N(t) = SW - z] dz - SW \int_{SW}^{\infty} f[N(t) = SW - z] dz .$$

As we noted in Chapter 3, the second term is nothing more than the product $SW\,P_{OUT}(SW)$. The derivations for $P_{OUT}(SW)$ were presented in Chapter 4 for the Normal modification. Therefore we can focus our attention on the derivations associated with the first part. We will denote this part as

$$B_1(SW) = \int_{SW}^{\infty} z f[N(t) = SW - z] dz . \tag{5.2}$$

Therefore, we can write

$$B(SW) = B_1(SW) - SW\,P_{OUT}(SW) . \tag{5.3}$$

B. Derivation of $B_1(SW)$

1. The case of $0 \le SW \le x_1$

For the case of $0 \le SW \le x_1$, when the net inventory probability distribution is given by equation (4.25) we can write $B_1(SW)$ as

$$
\begin{aligned}
B_1(SW) = {} & \frac{1}{Q_P Q_R} \int_{SW}^{x_1} z\left\{(z-\mu)\left[\Phi(\hat{0}) - \Phi(\hat{z})\right] - \sigma\left[\phi(\hat{0}) - \phi(\hat{z})\right]\right\} dz \\
& + \frac{1}{Q_P Q_R} \int_{x_1}^{x_2} z \left\{ \begin{array}{l} (z-\mu)\left[\Phi(\widehat{z-x_1}) - \Phi(\hat{z})\right] - \sigma\left[\phi(\widehat{z-x_1}) - \phi(\hat{z})\right] \\ + x_1\left[\Phi(\hat{0}) - \Phi(\widehat{z-x_1})\right] \end{array} \right\} dz \qquad (5.4) \\
& + \frac{1}{Q_P Q_R} \int_{x_2}^{\infty} z \left\{ \begin{array}{l} \dfrac{z-\mu}{Q_P Q_R}\left[\Phi(\widehat{z-x_1}) - \Phi(\hat{z})\right] - \dfrac{\sigma}{Q_P Q_R}\left[\phi(\widehat{z-x_1}) - \phi(\hat{z})\right] \\ + \dfrac{x_1}{Q_P Q_R}\left[\Phi(\widehat{z-x_2}) - \Phi(\widehat{z-x_1})\right] \\ + \dfrac{x_{Max} - z + \mu}{Q_P Q_R}\left[\Phi(\widehat{z - Min(z, x_{Max})}) - \Phi(\widehat{z-x_2})\right] \\ + \dfrac{\sigma}{Q_P Q_R}\left[\phi(\widehat{z - Min(z, x_{Max})}) - \phi(\widehat{z-x_2})\right] \end{array} \right\} dz .
\end{aligned}
$$

Next, we rewrite equation (5.4) into its individual component integrals.

$$
\begin{aligned}
Q_P Q_R B_1(SW) = & \int_{SW}^{x_1} z(z-\mu)\Phi(\hat{0})dz - \int_{SW}^{\infty} z(z-\mu)\Phi(\hat{z})dz - \sigma\phi(\hat{0})\int_{SW}^{x_1} zdz \\
& + \sigma\int_{SW}^{\infty} z\phi(\hat{z})dz + \int_{x_1}^{\infty} z(z-\mu)\Phi(\widehat{z-x_1})dz - \sigma\int_{x_1}^{\infty} z\phi(\widehat{z-x_1})dz \\
& + x_1\Phi(\hat{0})\int_{x_1}^{x_2} zdz - x_1\int_{x_1}^{\infty} z\Phi(\widehat{z-x_1})dz + x_1\int_{x_2}^{\infty} z\Phi(\widehat{z-x_2})dz \\
& + \int_{x_2}^{\infty} z(x_{Max}-z+\mu)\Phi(\widehat{z-Min(z,x_{Max})})dz \\
& - \int_{x_2}^{\infty} z(x_{Max}-z+\mu)\Phi(\widehat{z-x_2})dz \\
& + \sigma\int_{x_2}^{\infty} z\phi(\widehat{z-Min(z,x_{Max})})dz - \sigma\int_{x_2}^{\infty} z\phi(\widehat{z-x_2})dz.
\end{aligned} \tag{5.5}
$$

The first term of equation (5.5) is a constant.

$$
\begin{aligned}
\int_{SW}^{x_1} z(z-\mu)\Phi(\hat{0})dz &= \Phi(\hat{0})\int_{SW}^{x_1} z^2dz - \mu\Phi(\hat{0})\int_{SW}^{x_1} zdz \\
&= \Phi(\hat{0})\left[\frac{x_1^3}{3} - \frac{SW^3}{3}\right] - \mu\Phi(\hat{0})\left[\frac{x_1^2}{2} - \frac{SW^2}{2}\right] \\
&= \Phi(\hat{0})\left[\frac{x_1^3}{3} - \frac{SW^3}{3} - \frac{\mu x_1^2}{2} + \frac{\mu SW^2}{2}\right].
\end{aligned} \tag{5.6}
$$

For the second term of equation (5.5) we recall from Chapter 4 that

$$
\hat{z} = \frac{z-\mu}{\sigma}; \quad \text{so that} \quad z = \sigma\hat{z} + \mu, \quad z-\mu = \sigma\hat{z}, \quad \text{and} \quad dz = \sigma d\hat{z}.
$$

Therefore,

$$
\begin{aligned}
-\int_{SW}^{\infty} z(z-\mu)\Phi(\hat{z})dz &= -\sigma^2 \int_{SW}^{\infty} \left[\sigma\hat{z}^2 + \mu\hat{z}\right]\Phi(\hat{z})d\hat{z} \\
&= -\sigma^3 \int_{SW}^{\infty} \hat{z}^2\Phi(\hat{z})d\hat{z} - \sigma^2 \int_{SW}^{\infty} \hat{z}\Phi(\hat{z})d\hat{z}.
\end{aligned}
\tag{5.7}
$$

We now define $\delta(\hat{u})$ by using Identity 8 from Appendix 4 of Reference [3].

$$
\delta(\hat{u}) \equiv \int_{\hat{u}}^{\infty} \hat{r}^2\Phi(\hat{r})d\hat{r} \equiv \frac{1}{3}\left\{\left[\hat{u}^2 + 2\right]\phi(\hat{u}) - \hat{u}^3\Phi(\hat{u})\right\}.
\tag{5.8}
$$

Thus,

$$
\begin{aligned}
-\int_{SW}^{\infty} z(z-\mu)\Phi(\hat{z})dz &= -\sigma^3 \int_{SW}^{\infty} \hat{z}^2\Phi(\hat{z})d\hat{z} - \sigma^2\mu \int_{SW}^{\infty} \hat{z}\Phi(\hat{z})d\hat{z} \\
&= -\sigma^3\delta(\widehat{SW}) - \sigma^2\mu\gamma(\widehat{SW}),
\end{aligned}
\tag{5.9}
$$

where $\gamma(.)$ was defined in Chapter 4.

The third term of equation (5.5) is a constant.

$$
-\sigma\phi(\hat{0}) \int_{SW}^{x_1} z dz = -\sigma\phi(\hat{0})\left[\frac{x_1^2}{2} - \frac{SW^2}{2}\right].
\tag{5.10}
$$

The fourth term is

$$
\begin{aligned}
\sigma \int_{SW}^{\infty} z\phi(\hat{z})dz &= \sigma^2 \int_{SW}^{\infty} (\mu + \sigma\hat{z})\phi(\hat{z})d\hat{z} \\
&= \mu\sigma^2\Phi(\widehat{SW}) + \sigma^3\phi(\widehat{SW}).
\end{aligned}
\tag{5.11}
$$

Here we made use of Identity 1 of Appendix 4 of Reference [3]; namely,

$$
\int_{\hat{u}}^{\infty} \hat{r}\phi(\hat{r})d\hat{r} = \phi(\hat{u}).
$$

To reduce the fifth term of equation (5.5),

$$\int_{x_1}^{\infty} z(z-\mu)\Phi(\widehat{z-x_1})dz\,,$$

we define

$$\hat{u} = \frac{z-x_1-\mu}{\sigma}\,.$$

Then, $\hat{u} = \hat{0}$ *when* $z = x_1$, and

$$z = x_1 + \mu + \sigma\hat{u}\,;$$

$$z - \mu = x_1 + \sigma\hat{u}\,;$$

so that

$$\begin{aligned} z(z-\mu) &= (x_1+\sigma\hat{u})(x_1+\mu+\sigma\hat{u}) \\ &= x_1(x_1+\mu) + \sigma(2x_1+\mu)\hat{u} + \sigma^2\hat{u}^2\,. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_{x_1}^{\infty} z(z-\mu)\Phi(\widehat{z-x_1})dz \\ &\quad = \sigma x_1(x_1+\mu)\int_{\hat{0}}^{\infty}\Phi(\hat{u})d\hat{u} + \sigma^2(2x_1+\mu)\int_{\hat{0}}^{\infty}\hat{u}\Phi(\hat{u})d\hat{u} + \sigma^3\int_{\hat{0}}^{\infty}\hat{u}^2\Phi(\hat{u})d\hat{u} \qquad (5.12) \\ &\quad = \sigma x_1(x_1+\mu)\alpha(\hat{0}) + \sigma^2(2x_1+\mu)\gamma(\hat{0}) + \sigma^3\delta(\hat{0})\,. \end{aligned}$$

The sixth term uses the same definition for $\hat{u}$ as equation (5.12) did.

$$\begin{aligned} -\sigma\int_{x_1}^{\infty} z\phi(\widehat{z-x_1})dz &= -\sigma^2(x_1+\mu)\int_{\hat{0}}^{\infty}\phi(\hat{u})d\hat{u} - \sigma^3\int_{\hat{0}}^{\infty}\hat{u}\phi(\hat{u})d\hat{u} \qquad (5.13) \\ &= -\sigma^2(x_1+\mu)\Phi(\hat{0}) - \sigma^3\phi(\hat{0})\,. \end{aligned}$$

The seventh term is a constant term.

$$x_1\Phi(\hat{0})\int_{x_1}^{x_2} z dz = x_1\Phi(\hat{0})\left[\frac{x_2^2}{2} - \frac{x_1^2}{2}\right]. \tag{5.14}$$

The eighth term also uses the same identity for $\hat{u}$ as equation (5.12) did.

$$\begin{aligned} -x_1\int_{x_1}^{\infty} z\Phi(\widehat{z-x_1})dz &= -x_1(x_1+\mu)\int_{x_1}^{\infty}\Phi(\hat{u})dz - \sigma x_1\int_{x_1}^{\infty}\hat{u}\Phi(\hat{u})dz \\ &= -\sigma x_1(x_1+\mu)\int_{\hat{0}}^{\infty}\Phi(\hat{u})d\hat{u} - \sigma^2 x_1\int_{\hat{0}}^{\infty}\hat{u}\Phi(\hat{u})d\hat{u} \\ &= -\sigma x_1(x_1+\mu)\alpha(\hat{0}) - \sigma^2 x_1\gamma(\hat{0}). \end{aligned} \tag{5.15}$$

For the ninth term we define $\hat{u}$ as

$$\hat{u} = \frac{z - x_2 - \mu}{\sigma}.$$

Then, we get

$$\begin{aligned} x_1\int_{x_2}^{\infty} z\Phi(\widehat{z-x_2})dz &= \sigma x_1\int_{\hat{0}}^{\infty}(x_2+\mu+\sigma\hat{u})\Phi(\hat{u})d\hat{u} \\ &= \sigma x_1(x_2+\mu)\int_{\hat{0}}^{\infty}\Phi(\hat{u})d\hat{u} + \sigma^2 x_1\int_{\hat{0}}^{\infty}\hat{u}\Phi(\hat{u})d\hat{u} \\ &= \sigma x_1(x_2+\mu)\alpha(\hat{0}) + \sigma^2 x_1\gamma(\hat{0}). \end{aligned} \tag{5.16}$$

Notice that when we add equations (5.15) and (5.16) the sum is zero since the terms in one equation's reduced form cancel those in the other.

The tenth term uses

$$\hat{u} = \frac{z - x_{Max} - \mu}{\sigma}$$

for the second part (the integral from x_{Max} to ∞). The derivation of the reduced form of the tenth term is:

$$\int_{x_2}^{\infty} z(x_{Max} - z + \mu)\Phi(\widehat{z - Min(z, x_{Max})})dz$$

$$= \int_{x_2}^{x_{Max}} z(x_{Max} - z + \mu)\Phi(\hat{0})dz$$

$$+ \int_{x_{Max}}^{\infty} z(x_{Max} - z + \mu)\Phi(\widehat{z - x_{Max}})dz$$

$$= -\int_{x_2}^{x_{Max}} z^2\Phi(\hat{0})dz + (x_{Max} + \mu)\int_{x_2}^{x_{Max}} z\Phi(\hat{0})dz \tag{5.17}$$

$$- \sigma^2 \int_{\hat{0}}^{\infty} (x_{Max} + \mu + \sigma\hat{u})\hat{u}\Phi(\hat{u})d\hat{u}$$

$$= \Phi(\hat{0})\left\{-\frac{x_{Max}^3}{3} + \frac{x_2^3}{3} + (x_{Max} + \mu)\left[\frac{x_{Max}^2}{2} - \frac{x_2^2}{2}\right]\right\}$$

$$- \sigma^2(x_{Max} + \mu)\gamma(\hat{0}) - \sigma^3\delta(\hat{0}).$$

The eleventh term uses the same definition for $\hat{u}$ as equation (5.16).

$$-\int_{x_2}^{\infty} z(x_{Max} - z + \mu)\Phi(\widehat{z - x_2})dz$$

$$= -\sigma\int_{\hat{0}}^{\infty} (x_2 + \mu + \sigma\hat{u})(x_{Max} - x_2 - \sigma\hat{u})\Phi(\hat{u})d\hat{u}.$$

Expanding the integrand factors, we get

$$(x_2+\mu+\sigma\hat{u})(x_{Max}-x_2-\sigma\hat{u})$$
$$=\left[-\sigma^2\hat{u}^2-\sigma(x_2+\mu)\hat{u}+\sigma(x_{Max}-x_2)\hat{u}+(x_2+\mu)(x_{Max}-x_2)\right]$$
$$=\left[-\sigma^2\hat{u}^2-\sigma(x_2+\mu)\hat{u}+\sigma x_1\hat{u}+(x_2+\mu)x_1\right].$$

Therefore,

$$
\begin{aligned}
&-\int_{x_2}^{\infty} z(x_{Max}-z+\mu)\Phi(\widehat{z-x_2})dz \\
&=-\sigma\int_{\hat{0}}^{\infty}\left[-\sigma^2\hat{u}^2-\sigma(x_2+\mu)\hat{u}+\sigma x_1\hat{u}+(x_2+\mu)x_1\right]\Phi(\hat{u})d\hat{u} \\
&=\sigma^3\int_{\hat{0}}^{\infty}\hat{u}^2\Phi(\hat{u})d\hat{u}-\sigma^2(x_1-x_2-\mu)\int_{\hat{0}}^{\infty}\hat{u}\Phi(\hat{u})d\hat{u}-\sigma x_1(x_2+\mu)\int_{\hat{0}}^{\infty}\Phi(\hat{u})d\hat{u} \\
&=\sigma^3\delta(\hat{0})-\sigma^2(x_1-x_2-\mu)\gamma(\hat{0})-\sigma x_1(x_2+\mu)\alpha(\hat{0}).
\end{aligned}
\tag{5.18}
$$

The twelfth term uses the same definition of $\hat{u}$ as equation (5.17).

$$
\begin{aligned}
&\sigma\int_{x_2}^{\infty} z\phi(\widehat{z-Min(z,x_{Max})})dz \\
&=\sigma\int_{x_2}^{x_{max}} z\phi(\hat{0})dz+\sigma\int_{x_{Max}}^{\infty} z\phi(\widehat{z-x_{Max}})dz \\
&=\sigma\phi(\hat{0})\frac{z^2}{2}\Bigg|_{x_2}^{x_{Max}}+\sigma^2\int_{\hat{0}}^{\infty}(x_{Max}+\mu+\sigma\hat{u})\phi(\hat{u})d\hat{u} \\
&=\sigma\phi(\hat{0})\left[\frac{x_{Max}^2}{2}-\frac{x_2^2}{2}\right]+\sigma^2(x_{Max}+\mu)\Phi(\hat{0})+\sigma^3\phi(\hat{0}).
\end{aligned}
\tag{5.19}
$$

The final term of equation (5.5) is

$$-\sigma\int_{x_2}^{\infty} z\phi(\widehat{z-x_2})dz = -\sigma^2\int_{\hat{0}}^{\infty}(x_2+\mu+\sigma\hat{u})\phi(\hat{u})d\hat{u} \tag{5.20}$$
$$= -\sigma^2(x_2+\mu)\Phi(\hat{0}) - \sigma^3\phi(\hat{0})\,.$$

The final form of equation (5.5) involves summing all of the reduced forms contained in equations (5.6), and (5.9) through (5.20).

$$\begin{aligned}
Q_PQ_RB_1(SW) = {} & \Phi(\hat{0})\left[\frac{x_1^3}{3}-\frac{SW^3}{3}-\frac{\mu x_1^2}{2}+\frac{\mu SW^2}{2}\right]-\sigma^3\delta(\widehat{SW})-\sigma^2\mu\gamma(\widehat{SW}) \\
& -\sigma\phi(\hat{0})\left[\frac{x_1^2}{2}-\frac{SW^2}{2}\right]+\mu\sigma^2\Phi(\widehat{SW})+\sigma^3\phi(\widehat{SW}) \\
& +\sigma x_1(x_1+\mu)\alpha(\hat{0})+\sigma^2(2x_1+\mu)\gamma(\hat{0})+\sigma^3\delta(\hat{0}) \\
& -\sigma^2(x_1+\mu)\Phi(\hat{0})-\sigma^3\phi(\hat{0})+x_1\Phi(\hat{0})\left[\frac{x_2^2}{2}-\frac{x_1^2}{2}\right] \\
& -\sigma x_1(x_1+\mu)\alpha(\hat{0})-\sigma^2 x_1\gamma(\hat{0})+\sigma x_1(x_2+\mu)\alpha(\hat{0}) \\
& +\sigma^2 x_1\gamma(\hat{0})+\Phi(\hat{0})\left\{-\frac{x_{Max}^3}{3}+\frac{x_2^3}{3}+(x_{Max}+\mu)\left[\frac{x_{Max}^2}{2}-\frac{x_2^2}{2}\right]\right\} \\
& -\sigma^2(x_{Max}+\mu)\gamma(\hat{0})-\sigma^3\delta(\hat{0})+\sigma^3\delta(\hat{0})-\sigma^2(x_1-x_2-\mu)\gamma(\hat{0}) \\
& -\sigma x_1(x_2+\mu)\alpha(\hat{0})+\sigma\phi(\hat{0})\left[\frac{x_{Max}^2}{2}-\frac{x_2^2}{2}\right]+\sigma^2(x_{Max}+\mu)\Phi(\hat{0}) \\
& +\sigma^3\phi(\hat{0})-\sigma^2(x_2+\mu)\Phi(\hat{0})-\sigma^3\phi(\hat{0})\,.
\end{aligned} \tag{5.21}$$

Equation (5.21) can be substantially reduced by collecting all of the like terms. Many then cancel each other. The resulting equation for $B_1(SW)$ for the case where $0 \le SW \le x_1$ is therefore:

$$B_1(SW)=\frac{\Phi(\hat{0})}{Q_PQ_R}\left[\frac{x_{Max}^3}{6}-\frac{x_2^3}{6}-\frac{x_1^3}{6}-\frac{SW^3}{3}+\frac{\mu x_{Max}^2}{2}-\frac{\mu x_2^2}{2}-\frac{\mu x_1^2}{2}+\frac{\mu SW^2}{2}\right]$$
$$+\frac{\sigma\phi(\hat{0})}{Q_PQ_R}\left[\frac{x_{Max}^2}{2}-\frac{x_2^2}{2}-\frac{x_1^2}{2}+\frac{SW^2}{2}\right] \tag{5.22}$$
$$+\frac{\sigma^2}{Q_PQ_R}\left\{\begin{array}{l}-\sigma\left[\delta(\widehat{SW})-\phi(\widehat{SW})\right]-\mu\left[\gamma(\widehat{SW})-\Phi(\widehat{SW})\right]\\ +\sigma\left[\delta(\hat{0})-\phi(\hat{0})\right]+\mu\left[\gamma(\hat{0})-\Phi(\hat{0})\right]\end{array}\right\}.$$

2. The case of $x_1 \le SW \le x_2$

Equation (5.4) changes to the following form for this case since its first line no longer applies.

$$B_1(SW)=\frac{1}{Q_PQ_R}\int_{SW}^{x_2} z\left\{\begin{array}{l}(z-\mu)\left[\Phi(\widehat{z-x_1})-\Phi(\hat{z})\right]-\sigma\left[\phi(\widehat{z-x_1})-\phi(\hat{z})\right]\\ +x_1\left[\Phi(\hat{0})-\Phi(\widehat{z-x_1})\right]\end{array}\right\}dz$$
$$+\frac{1}{Q_PQ_R}\int_{x_2}^{\infty} z\left\{\begin{array}{l}\frac{z-\mu}{Q_PQ_R}\left[\Phi(\widehat{z-x_1})-\Phi(\hat{z})\right]-\frac{\sigma}{Q_PQ_R}\left[\phi(\widehat{z-x_1})-\phi(\hat{z})\right]\\ +\frac{x_1}{Q_PQ_R}\left[\Phi(\widehat{z-x_2})-\Phi(\widehat{z-x_1})\right]\\ +\frac{x_{Max}-z+\mu}{Q_PQ_R}\left[\Phi(\widehat{z-Min(z,x_{Max})})-\Phi(\widehat{z-x_2})\right]\\ +\frac{\sigma}{Q_PQ_R}\left[\phi(\widehat{z-Min(z,x_{Max})})-\phi(\widehat{z-x_2})\right]\end{array}\right\}dz. \tag{5.23}$$

Therefore, equation (5.5) now changes to the following form. Note that the x_1 lower bounds on the integrals in equation (5.5) have been replaced by SW.

$$
\begin{aligned}
Q_P Q_R B_1(SW) = &- \int_{SW}^{\infty} z(z-\mu)\Phi(\hat{z})dz + \sigma \int_{SW}^{\infty} z\phi(\hat{z})dz \\
&+ \int_{SW}^{\infty} z(z-\mu)\Phi(\widehat{z-x_1})dz - \sigma \int_{SW}^{\infty} z\phi(\widehat{z-x_1})dz \\
&+ x_1\Phi(\hat{0}) \int_{SW}^{x_2} zdz - x_1 \int_{SW}^{\infty} z\Phi(\widehat{z-x_1})dz + x_1 \int_{x_2}^{\infty} z\Phi(\widehat{z-x_2})dz \\
&+ \int_{x_2}^{\infty} z(x_{Max} - z + \mu)\Phi(\widehat{z - Min(z, x_{Max})})dz \\
&- \int_{x_2}^{\infty} z(x_{Max} - z + \mu)\Phi(\widehat{z-x_2})dz \\
&+ \sigma \int_{x_2}^{\infty} z\phi(\widehat{z - Min(z, x_{Max})})dz - \sigma \int_{x_2}^{\infty} z\phi(\widehat{z-x_2})dz.
\end{aligned}
\tag{5.24}
$$

The reduced forms for the first and last three lines remain the same as for the preceding case. Thus, we need only derive the reduced forms for the second and third lines. Three terms in the two lines involve $\hat{u} = (\widehat{z-x_1}) = \dfrac{z-x_1-\mu}{\sigma}$. Therefore, in these terms we will use

$$z = x_1 + \mu + \sigma\hat{u};$$

$$z - \mu = x_1 + \sigma\hat{u};$$

and, when $z = SW$, $\hat{u} = (\widehat{SW - x_1})$. The derivations of the reduced forms for the three terms are shown below.

$$\int_{SW}^{\infty} z(z-\mu)\Phi(\widehat{z-x_1})dz = \sigma \int_{(\widehat{SW-x_1})}^{\infty} (x_1+\mu+\sigma\hat{u})(x_1+\sigma\hat{u})\Phi(\hat{u})d\hat{u}$$

$$= \sigma x_1(x_1+\mu) \int_{(\widehat{SW-x_1})}^{\infty} \Phi(\hat{u})d\hat{u} + \sigma^2(2x_1+\mu) \int_{(\widehat{SW-x_1})}^{\infty} \hat{u}\Phi(\hat{u})d\hat{u} \qquad (5.25)$$

$$+\sigma^3 \int_{(\widehat{SW-x_1})}^{\infty} \hat{u}^2\Phi(\hat{u})d\hat{u}$$

$$= \sigma x_1(x_1+\mu)\alpha(\widehat{SW-x_1}) + \sigma^2(2x_1+\mu)\gamma(\widehat{SW-x_1}) + \sigma^3\delta(\widehat{SW-x_1}).$$

$$-\sigma \int_{SW}^{\infty} z\phi(\widehat{z-x_1})dz = -\sigma^2 \int_{\widehat{SW-x_1}}^{\infty} (x_1+\mu+\sigma\hat{u})\phi(\hat{u})d\hat{u}$$

$$= -\sigma^2(x_1+\mu) \int_{\widehat{SW-x_1}}^{\infty} \phi(\hat{u})d\hat{u} - \sigma^3 \int_{\widehat{SW-x_1}}^{\infty} \hat{u}\phi(\hat{u})d\hat{u} \qquad (5.26)$$

$$= -\sigma^2(x_1+\mu)\Phi(\widehat{SW-x_1}) - \sigma^3\phi(\widehat{SW-x_1}).$$

$$-x_1 \int_{SW}^{\infty} z\Phi(\widehat{z-x_1})dz = -\sigma x_1 \int_{\widehat{SW-x_1}}^{\infty} (x_1+\mu+\sigma\hat{u})\phi(\hat{u})d\hat{u} \qquad (5.27)$$

$$= -\sigma x_1(x_1+\mu)\alpha(\widehat{SW-x_1}) - \sigma^2 x_1\gamma(\widehat{SW-x_1}).$$

The remaining term is a constant.

$$x_1\Phi(\hat{0}) \int_{SW}^{x_2} zdz = x_1\Phi(\hat{0})\left[\frac{x^2}{2} - \frac{SW^2}{2}\right]. \qquad (5.28)$$

Adding all of the reduced terms for equation (5.24) gives

$$
\begin{aligned}
Q_P Q_R B_1(SW) = & -\sigma^3\delta(\widehat{SW}) - \sigma^2\mu\gamma(\widehat{SW}) \\
& + \mu\sigma^2\Phi(\widehat{SW}) + \sigma^3\phi(\widehat{SW}) \\
& + \sigma x_1(x_1+\mu)\alpha(\widehat{SW-x_1}) + \sigma^2(2x_1+\mu)\gamma(\widehat{SW-x_1}) \\
& + \sigma^3\delta(\widehat{SW-x_1}) - \sigma^2(x_1+\mu)\Phi(\widehat{SW-x_1}) \\
& - \sigma^3\phi(\widehat{SW-x_1}) + x_1\Phi(\hat{0})\left[\frac{x_2^2}{2} - \frac{SW^2}{2}\right] \\
& - \sigma x_1(x_1+\mu)\alpha(\widehat{SW-x_1}) - \sigma^2 x_1\gamma(\widehat{SW-x_1}) + \sigma x_1(x_2+\mu)\alpha(\hat{0}) \\
& + \sigma^2 x_1\gamma(\hat{0}) + \Phi(\hat{0})\left\{-\frac{x_{Max}^3}{3} + \frac{x_2^3}{3} + (x_{Max}+\mu)\left[\frac{x_{Max}^2}{2} - \frac{x_2^2}{2}\right]\right\} \\
& - \sigma^2(x_{Max}+\mu)\gamma(\hat{0}) - \sigma^3\delta(\hat{0}) + \sigma^3\delta(\hat{0}) - \sigma^2(x_1 - x_2 - \mu)\gamma(\hat{0}) \\
& - \sigma x_1(x_2+\mu)\alpha(\hat{0}) + \sigma\phi(\hat{0})\left[\frac{x_{Max}^2}{2} - \frac{x_2^2}{2}\right] + \sigma^2(x_{Max}+\mu)\Phi(\hat{0}) \\
& + \sigma^3\phi(\hat{0}) - \sigma^2(x_2+\mu)\Phi(\hat{0}) - \sigma^3\phi(\hat{0}).
\end{aligned}
\tag{5.29}
$$

Again, after collecting like terms, a substantially reduction in equation (5.29) occurs. Thus, for the case of $x_1 < SW \le x_2$ we get, as the final form for $B_1(SW)$,

$$
\begin{aligned}
B_1(SW) = & \frac{\Phi(\hat{0})}{Q_P Q_R}\left[\frac{x_{Max}^3}{6} - \frac{x_2^3}{6} + \frac{\mu x_{Max}^2}{2} - \frac{\mu x_2^2}{2} - \frac{x_1 SW^2}{2}\right] \\
& + \frac{\sigma\phi(\hat{0})}{Q_P Q_R}\left[\frac{x_{Max}^2}{2} - \frac{x_2^2}{2}\right] \\
& + \frac{\sigma^2}{Q_P Q_R}\left\{\begin{aligned}
& -\sigma\left[\delta(\widehat{SW}) - \phi(\widehat{SW})\right] \\
& -\mu\left[\gamma(\widehat{SW}) - \Phi(\widehat{SW})\right] \\
& +\sigma\left[\delta(\widehat{SW-x_1}) - \phi(\widehat{SW-x_1})\right] \\
& +\mu\left[\gamma(\widehat{SW-x_1}) - \Phi(\widehat{SW-x_1})\right] \\
& -x_1\left[\gamma(\hat{0}) - \Phi(\hat{0})\right]
\end{aligned}\right\}.
\end{aligned}
\tag{5.30}
$$

3. **The case of $x_2 \le SW \le x_{Max}$**

Only the third part of equation (5.4) applies for this case.

$$B_1(SW) = \frac{1}{Q_P Q_R} \int_{SW}^{\infty} z \left\{ \begin{array}{l} \frac{z-\mu}{Q_P Q_R}\left[\Phi(\widehat{z-x_1}) - \Phi(\hat{z})\right] - \frac{\sigma}{Q_P Q_R}\left[\phi(\widehat{z-x_1}) - \phi(\hat{z})\right] \\ + \frac{x_1}{Q_P Q_R}\left[\Phi(\widehat{z-x_2}) - \Phi(\widehat{z-x_1})\right] \\ + \frac{x_{Max}-z+\mu}{Q_P Q_R}\left[\Phi(\widehat{z-Min(z,x_{Max})}) - \Phi(\widehat{z-x_2})\right] \\ + \frac{\sigma}{Q_P Q_R}\left[\phi(\widehat{z-Min(z,x_{Max})}) - \phi(\widehat{z-x_2})\right] \end{array} \right\} dz \,. \quad (5.31)$$

As a consequence, equation (5.5) is reduced to

$$\begin{aligned} Q_P Q_R B_1(SW) = & -\int_{SW}^{\infty} z(z-\mu)\Phi(\hat{z})dz + \sigma \int_{SW}^{\infty} z\phi(\hat{z})dz \\ & + \int_{SW}^{\infty} z(z-\mu)\Phi(\widehat{z-x_1})dz - \sigma \int_{SW}^{\infty} z\phi(\widehat{z-x_1})dz \\ & - x_1 \int_{SW}^{\infty} z\Phi(\widehat{z-x_1})dz + x_1 \int_{SW}^{\infty} z\Phi(\widehat{z-x_2})dz \\ & + \int_{SW}^{\infty} z(x_{Max}-z+\mu)\Phi(\widehat{z-Min(z,x_{Max})})dz \\ & - \int_{SW}^{\infty} z(x_{Max}-z+\mu)\Phi(\widehat{z-x_2})dz \\ & + \sigma \int_{SW}^{\infty} z\phi(\widehat{z-Min(z,x_{Max})})dz - \sigma \int_{SW}^{\infty} z\phi(\widehat{z-x_2})dz \,. \end{aligned} \quad (5.32)$$

The first five terms will not change for this case. We will first focus on the terms involving $(\widehat{z-x_2})$. In those terms we define $\hat{u}$ as $\hat{u} = (\widehat{z-x_2}) = \frac{z-x_2-\mu}{\sigma}$ so that $z = x_2 - \mu - \sigma\hat{u}$.

$$x_1 \int_{SW}^{\infty} z\Phi(\widehat{z-x_2})dz = \sigma x_1 \int_{\widehat{SW-x_2}}^{\infty} (x_2 + \mu + \sigma\hat{u})\Phi(\hat{u})d\hat{u}$$

$$= \sigma x_1(x_2 + \mu)\alpha(\widehat{SW - x_2}) + \sigma^2 x_1 \gamma(\widehat{SW - x_2})\,. \tag{5.33}$$

The next term,

$$-\int_{SW}^{\infty} z(x_{Max} - z + \mu)\Phi(\widehat{z-x_2})dz = -\sigma \int_{\widehat{SW-x_2}}^{\infty} (x_2 + \mu + \sigma\hat{u})(x_{Max} - x_2 - \mu)\Phi(\hat{u})d\hat{u}\,,$$

has a long integrand when the factor terms are multiplied together. Doing that first,

$$(x_2 + \mu + \sigma\hat{u})(x_{Max} - x_2 - \sigma\hat{u}) = x_1(x_2 + \mu) + \sigma(x_1 - x_2 + \mu)\hat{u} - \sigma^2\hat{u}^2\,.$$

Therefore,

$$-\int_{SW}^{\infty} z(x_{Max} - z + \mu)\Phi(\widehat{z - x_2})dz$$

$$= -\sigma x_1(x_2 + \mu) \int_{\widehat{SW-x_2}}^{\infty} \Phi(\hat{u})d\hat{u}$$

$$-\sigma^2(x_1 - x_2 - \mu) \int_{\widehat{SW-x_2}}^{\infty} \hat{u}\Phi(\hat{u})d\hat{u}$$

$$+\sigma^3 \int_{\widehat{SW-x_2}}^{\infty} \hat{u}^2\Phi(\hat{u})d\hat{u}$$

$$= -\sigma x_1(x_2 + \mu)\alpha(\widehat{SW - x_2}) \tag{5.34}$$

$$-\sigma^2(x_1 - x_2 - \mu)\gamma(\widehat{SW - x_2})$$

$$+\sigma^3\delta(\widehat{SW - x_2})\,.$$

The final term involving $(\widehat{z - x_2})$ is

$$-\sigma \int_{SW}^{\infty} z\phi(\widehat{z-x_2})dz = -\sigma^2 \int_{\widehat{SW-x_2}}^{\infty} (x_2+\mu+\sigma\hat{u})\phi(\hat{u})d\hat{u}$$
$$= -\sigma^2(x_2+\mu)\Phi(\widehat{SW-x_2}) - \sigma^3\phi(\widehat{SW-x_2}). \tag{5.35}$$

The remaining two terms need to be subdivided in the intervals $SW \le z \le x_{Max}$ and $z \ge x_{Max}$. In the latter range, $\hat{u} = \dfrac{z - x_{Max} - \mu}{\sigma}$.

$$\int_{SW}^{\infty} z(x_{Max} - z + \mu)\Phi(\widehat{z - Min(z, x_{Max})})dz$$
$$= \int_{SW}^{x_{max}} z(x_{Max} - z + \mu)\Phi(\hat{0})dz + \int_{x_{Max}}^{\infty} z(x_{Max} - z + \mu)\Phi(\widehat{z - x_{Max}})dz$$
$$= \Phi(\hat{0})\int_{SW}^{x_{max}} \left[z(x_{Max}+\mu) - z^2\right]dz - \sigma^2 \int_{\hat{0}}^{\infty}(x_{Max}+\mu+\sigma\hat{u})\hat{u}\Phi(\hat{u})d\hat{u}$$
$$= \Phi(\hat{0})\left\{(x_{Max}+\mu)\left[\frac{x_{Max}^2}{2} - \frac{SW^2}{2}\right] - \frac{x_{Max}^3}{3} + \frac{SW^3}{3}\right\} \tag{5.36}$$
$$- \sigma^2(x_{Max}+\mu)\int_{\hat{0}}^{\infty}\hat{u}\Phi(\hat{u})d\hat{u} - \sigma^3\int_{\hat{0}}^{\infty}\hat{u}^2\Phi(\hat{u})d\hat{u}$$
$$= \Phi(\hat{0})\left\{\frac{x_{Max}^3}{6} - \frac{(x_{Max}+\mu)SW^2}{2} + \frac{SW^3}{3} + \frac{\mu x_{Max}^2}{2}\right\}$$
$$- \sigma^2(x_{Max}+\mu)\gamma(\hat{0}) - \sigma^3\delta(\hat{0}).$$

$$\sigma\int_{SW}^{\infty} z\phi(\widehat{z - Min(z, x_{Max})})dz = \sigma\phi(\hat{0})\int_{SW}^{x_{Max}} z dz + \sigma^2\int_{\hat{0}}^{\infty}(x_{Max}+\mu+\sigma\hat{u})\phi(\hat{u})d\hat{u}$$
$$= \sigma\phi(\hat{0})\left[\frac{x_{Max}^2}{2} - \frac{SW^2}{2}\right] + \sigma^2(x_{Max}+\mu)\Phi(\hat{0}) + \sigma^3\phi(\hat{0}). \tag{5.37}$$

Adding up all of the reduced terms for equation (5.32) gives

$$
\begin{aligned}
Q_P Q_R B_1(SW) = &-\sigma^3\delta(\widehat{SW}) - \sigma^2\mu\gamma(\widehat{SW}) + \mu\sigma^2\Phi(\widehat{SW}) + \sigma^3\phi(\widehat{SW}) \\
&+ \sigma x_1(x_1+\mu)\alpha(\widehat{SW-x_1}) + \sigma^2(2x_1+\mu)\gamma(\widehat{SW-x_1}) \\
&+ \sigma^3\delta(\widehat{SW-x_1}) - \sigma^2(x_1+\mu)\Phi(\widehat{SW-x_1}) \\
&- \sigma^3\phi(\widehat{SW-x_1}) - \sigma x_1(x_1+\mu)\alpha(\widehat{SW-x_1}) - \sigma^2 x_1\gamma(\widehat{SW-x_1}) \\
&+ \sigma x_1(x_2+\mu)\alpha(\widehat{SW-x_2}) + \sigma^2 x_1\gamma(\widehat{SW-x_2}) \\
&- \sigma x_1(x_2+\mu)\alpha(\widehat{SW-x_2}) - \sigma^2(x_1-x_2-\mu)\gamma(\widehat{SW-x_2}) \\
&+ \sigma^3\delta(\widehat{SW-x_2}) - \sigma^2(x_2+\mu)\Phi(\widehat{SW-x_2}) - \sigma^3\phi(\widehat{SW-x_2}) \\
&+ \Phi(\hat{0})\left\{\frac{x_{Max}^3}{6} - \frac{(x_{Max}+\mu)SW^2}{2} + \frac{SW^3}{3} + \frac{\mu x_{Max}^2}{2}\right\} \\
&- \sigma^2(x_{Max}+\mu)\gamma(\hat{0}) - \sigma^3\delta(\hat{0}) + \sigma\phi(\hat{0})\left[\frac{x_{Max}^2}{2} - \frac{SW^2}{2}\right] \\
&+ \sigma^2(x_{Max}+\mu)\Phi(\hat{0}) + \sigma^3\phi(\hat{0}).
\end{aligned}
\tag{5.38}
$$

Collecting common terms results in the final version of $B_1(SW)$ for the case of $x_2 < SW \le x_{Max}$.

$$
\begin{aligned}
B_1(SW) = &\frac{\Phi(\hat{0})}{Q_P Q_R}\left[\frac{x_{Max}^3}{6} + \frac{SW^3}{3} + \frac{\mu x_{Max}^2}{2} - \frac{(x_{Max}+\mu)SW^2}{2}\right] \\
&+ \frac{\sigma\phi(\hat{0})}{Q_P Q_R}\left[\frac{x_{Max}^2}{2} - \frac{x_2^2}{2}\right] \\
&+ \frac{\sigma^2}{Q_P Q_R}\left\{
\begin{array}{l}
-\sigma\left[\delta(\widehat{SW}) - \phi(\widehat{SW})\right] \\
-\mu\left[\gamma(\widehat{SW}) - \Phi(\widehat{SW})\right] \\
+\sigma\left[\delta(\widehat{SW-x_1}) - \phi(\widehat{SW-x_1})\right] \\
+\mu\left[\gamma(\widehat{SW-x_1}) - \Phi(\widehat{SW-x_1})\right] \\
+\sigma\left[\delta(\widehat{SW-x_2}) - \phi(\widehat{SW-x_2})\right] \\
+\mu\left[\gamma(\widehat{SW-x_2}) - \Phi(\widehat{SW-x_2})\right]
\end{array}
\right\}.
\end{aligned}
\tag{5.39}
$$

4. The case of $SW \geq x_{Max}$

Equations (5.31) and (5.32) apply here also. The part that needs no longer to be considered is where $Min(z,x_{Max}) = z$ which was the case when $SW \leq x_{Max}$. Thus, equation (5.31) can now be written as

$$B_1(SW) = \frac{1}{Q_P Q_R}\int_{SW}^{\infty} z \left\{ \begin{array}{l} \frac{z-\mu}{Q_P Q_R}\left[\Phi(\widehat{z-x_1}) - \Phi(\hat{z})\right] - \frac{\sigma}{Q_P Q_R}\left[\phi(\widehat{z-x_1}) - \phi(\hat{z})\right] \\ + \frac{x_1}{Q_P Q_R}\left[\Phi(\widehat{z-x_2}) - \Phi(\widehat{z-x_1})\right] \\ + \frac{x_{Max}-z+\mu}{Q_P Q_R}\left[\Phi(\widehat{z-x_{Max}}) - \Phi(\widehat{z-x_2})\right] \\ + \frac{\sigma}{Q_P Q_R}\left[\phi(\widehat{z-x_{Max}}) - \phi(\widehat{z-x_2})\right] \end{array} \right\} dz \,. \quad (5.40)$$

Therefore, equation (5.32) can be rewritten as

$$\begin{aligned} Q_P Q_R B_1(SW) = &-\int_{SW}^{\infty} z(z-\mu)\Phi(\hat{z})dz + \sigma\int_{SW}^{\infty} z\phi(\hat{z})dz \\ &+ \int_{SW}^{\infty} z(z-\mu)\Phi(\widehat{z-x_1})dz - \sigma\int_{SW}^{\infty} z\phi(\widehat{z-x_1})dz \\ &- x_1\int_{SW}^{\infty} z\Phi(\widehat{z-x_1})dz + x_1\int_{SW}^{\infty} z\Phi(\widehat{z-x_2})dz \\ &+ \int_{SW}^{\infty} z(x_{Max}-z+\mu)\Phi(\widehat{z-x_{Max}}))dz \\ &- \int_{SW}^{\infty} z(x_{Max}-z+\mu)\Phi(\widehat{z-x_2})dz \\ &+ \sigma\int_{SW}^{\infty} z\phi(\widehat{z-x_{Max}}))dz - \sigma\int_{SW}^{\infty} z\phi(\widehat{z-x_2})dz \,. \end{aligned} \quad (5.41)$$

Only two terms need to be reduced. These are the term which is the entire fourth line and the first term of the sixth line. All of the rest have been reduced above.

$$\int_{SW}^{\infty} z(x_{Max} - z + \mu)\Phi(\widehat{z - x_{Max}})dz$$
$$= -\sigma^2 \int_{\widehat{SW - x_{Max}}}^{\infty} (x_{Max} + \mu + \sigma\hat{u})\hat{u}\Phi(\hat{u})d\hat{u} \tag{5.42}$$
$$= -\sigma^2(x_{Max} + \mu)\gamma(\widehat{SW - x_{\max}}) - \sigma^3\delta(SW - x_{Max}).$$

$$\sigma\int_{SW}^{\infty} z\phi(\widehat{z - x_{Max}})dz = \sigma \int_{\widehat{SW - x_{Max}}}^{\infty} (x_{Max} + \mu + \sigma\hat{u})\phi(\hat{u})d\hat{u}$$
$$= \sigma(x_{Max} + \mu)\int_{\widehat{SW - x_{Max}}}^{\infty} \phi(\hat{u})d\hat{u} + \sigma^2 \int_{\widehat{SW - x_{Max}}}^{\infty} \hat{u}\phi(\hat{u})d\hat{u} \tag{5.43}$$
$$= \sigma(x_{Max} + \mu)\Phi(\widehat{SW - x_{Max}}) + \sigma^2\phi(\widehat{SW - x_{Max}}).$$

Adding all of the reduced terms for this case gives

$$\begin{aligned}
Q_P Q_R B_1(SW) = &-\sigma^3\delta(\widehat{SW}) - \sigma^2\mu\gamma(\widehat{SW}) + \mu\sigma^2\Phi(\widehat{SW}) + \sigma^3\phi(\widehat{SW}) \\
&+ \sigma x_1(x_1 + \mu)\alpha(\widehat{SW - x_1}) + \sigma^2(2x_1 + \mu)\gamma(\widehat{SW - x_1}) \\
&+ \sigma^3\delta(\widehat{SW - x_1}) - \sigma^2(x_1 + \mu)\Phi(\widehat{SW - x_1}) \\
&- \sigma^3\phi(\widehat{SW - x_1}) - \sigma x_1(x_1 + \mu)\alpha(\widehat{SW - x_1}) - \sigma^2 x_1\gamma(\widehat{SW - x_1}) \\
&+ \sigma x_1(x_2 + \mu)\alpha(\widehat{SW - x_2}) + \sigma^2 x_1\gamma(\widehat{SW - x_2}) \\
&- \sigma x_1(x_2 + \mu)\alpha(\widehat{SW - x_2}) - \sigma^2(x_1 - x_2 - \mu)\gamma(\widehat{SW - x_2}) \\
&+ \sigma^3\delta(\widehat{SW - x_2}) - \sigma^2(x_2 + \mu)\Phi(\widehat{SW - x_2}) - \sigma^3\phi(\widehat{SW - x_2}) \\
&+ -\sigma^2(x_{Max} + \mu)\gamma(\widehat{SW - x_{\max}}) - \sigma^3\delta(SW - x_{Max}) \\
&+ \sigma(x_{Max} + \mu)\Phi(\widehat{SW - x_{Max}}) + \sigma^2\phi(\widehat{SW - x_{Max}}).
\end{aligned} \tag{5.44}$$

Collecting like terms then gives the final form for $B_1(SW)$ for the case of $SW \geq x_{Max}$.

$$B_1(SW)=\frac{\sigma^2}{Q_PQ_R}\left\{\begin{array}{l}-\sigma\left[\delta(\widetilde{SW})-\phi(\widetilde{SW})\right]-\mu\left[\gamma(\widetilde{SW})-\Phi(\widetilde{SW})\right]\\+\sigma\left[\delta(\widetilde{SW-x_1})-\phi(\widetilde{SW-x_1})\right]\\+\mu\left[\gamma(\widetilde{SW-x_1})-\Phi(\widetilde{SW-x_1})\right]\\+\sigma\left[\delta(\widetilde{SW-x_2})-\phi(\widetilde{SW-x_2})\right]\\+\mu\left[\gamma(\widetilde{SW-x_2})-\Phi(\widetilde{SW-x_2})\right]\\-\sigma\left[\delta(\widetilde{SW-x_{Max}})-\phi(\widetilde{SW-x_{Max}})\right]\\-(x_{Max}+\mu)\left[\gamma(\widetilde{SW-x_{Max}})-\Phi(\widetilde{SW-x_{Max}})\right]\end{array}\right\}. \quad (5.45)$$

C. Summary

This completes the derivations of the formulas for $B_1(SW)$. In review, when $0 \le SW \le x_1$ then $B_1(SW)$ is given by equation (5.22); when $x_1 \le SW \le x_2$, $B_1(SW)$ is given by equation (5.30); when $x_2 \le SW \le x_{Max}$, $B_1(SW)$ is given by equation (5.39); and when $SW \ge x_{Max}$, $B_1(SW)$ is given by equation (5.45). To get $B(SW)$ we use equation (5.3); namely,

$$B(SW)=B_1(SW)-SW\,P_{OUT}(SW).$$

The equations for $P_{OUT}(SW)$ were presented in Chapter 4.

CHAPTER 6 - MODELS FOR ESTIMATING SAFETY STOCK

A. Introduction

Hadley and Whitin on page 165 of Reference [3] state that safety stock is "by definition, the expected net inventory at the time of arrival of a procurement." Their definition is, of course, limited in applicability to only a consumable item where replenishment of stock is by procurement. However, a natural extension of this definition to a repairable item is possible when such an item is always successful repaired after failure. For such an item there are never any procurements and safety stock can therefore be defined as "the expected net inventory at the time of the arrival of a batch of repaired item."

Considering the pure procurement case first, the reorder point R can be defined as

$$R = D * PCLT + SS_P$$

where

D= expected quarterly demand;

$PCLT$ = procurement lead time, quarters;

SS_P = procurement safety stock;

and $D * PCLT$ represents the expected demand during procurement lead time.

Rewriting this equation gives

$$SS_P = R - D * PCLT.$$

We are concerned about representing SS_P in terms of the maximum inventory position for a readiness-base repairable item inventory model. As in other chapters in this report we will use SW to represent the maximum inventory

position for the readiness-based sparing models. From Hadley and Whitin [3], page 181, we know that

$$IP_{Max} = SW = R + Q_P$$

for a consumable item (to be called hereafter "the pure procurement case"). Therefore,

$$SS_P = SW - D * PCLT - Q_P \, . \tag{6.1}$$

Similarly, if we have the pure repair case with batched repair of quantity Q_R then

$$SS_R = SW - D * RTAT - Q_R; \tag{6.2}$$

if $RTAT$ is the batch repair time.

If, on the other hand, we have a batch of Q_R carcasses being repaired where each unit returns to inventory from the repair depot as soon as it completes repair and there is a time between inductions of units into the repair process, denoted by REP, we have a special case of Baker's model [1] presented in his Chapter IV (Non-Instantaneous Repair Assessment) and

$$SS_R = SW - D\left[RTAT + \left(\frac{Q_R - 1}{2}\right)REP\right] - \frac{Q_R}{2} \, . \tag{6.3}$$

Here the average repair turnaround time is

$$RTAT + \left(\frac{Q_R - 1}{2}\right)REP,$$

and the average number of units repaired over that time is $\frac{Q_R}{2}$.

Equations (6.1), (6.2), and (6.3) have been confirmed through simulation runs made in the process of developing a generalized safety stock model. The definition for the generalized safety stock was described in Maher [4].

It is "the expected net inventory at the time of arrival of a procurement and/or the arrival of a repaired carcass from the repair depot."

The question of what form the generalized safety stock equation should take is not obvious. For a general repairable item some repairs are successful and some procurements are needed to replenishment the attritions due to unsuccessful repair or a carcass not being returned for repair by the customer when he demands a functioning unit to replace the failed one. Because both procurements of new units to replace old units which could not be repaired and successful repair of other old ones contribute to the ready-for-issue (RFI) inventory, the effect of the carcass return rate (CRR) from the customers and the repair survival rate (RSR) of the repair process should play an important role in the generation of the general safety stock formula. A first thought was that the formula should be a convex combination of the pure procurement safety stock formula and the pure repair safety stock formula where the product $CRR * RSR$ would provide the weighting factor; namely,

$$SS = (1 - CRR * RSR)SS_P + CRR * RSR * SS_R; \tag{6.4}$$

since $0 \le CRR \le 1.0$, $0 \le RSR \le 1.0$ and $CRR = RSR = 0$ for pure procurement, $CRR = RSR = 1.0$ for pure repair. Equation (6.4) results in

$$\begin{aligned} SS &= (1 - CRR * RSR)(SW - D * PCLT - Q_P) \\ &\qquad + CRR * RSR(SW - D * RTAT - Q_R) \\ &= SW - (1 - CRR * RSR)D * PCLT \\ &\qquad - CRR * RSR * D * RTAT \\ &\qquad - (1 - CRR * RSR)Q_P - CRR * RSR * Q_R \\ &= SW - PPV - \left[(1 - CRR * RSR)Q_P + CRR * RSR * Q_R\right]; \end{aligned} \tag{6.5}$$

where

$$PPV \equiv (1 - CRR * RSR)D * PCLT + CRR * RSR * D * RTAT \,. \tag{6.6}$$

PPV stands for the "Program Problem Variable," the name used by the NAVICP for the expected demand during the aggregate lead time L_2, where

$$L_2 \equiv (1 - CRR * RSR)PCLT + CRR * RSR * RTAT. \qquad (6.7)$$

In the case where we allow each item entering repair to wait a time REP until its predecessor has passed through the first workstation, we get the modified form for PPV due to Baker [1]: namely,

$$ZB \equiv (1 - CRR * RSR)D * PCLT + CRR * RSR * D\left[RTAT + \frac{Q_R - 1}{2} REP\right]. \qquad (6.8)$$

Equation (6.4) then takes on the following form based on equations (6.1) and (6.3).

$$SS = SW - ZB - \left[(1 - CRR * RSR)Q_P + CRR * RSR * \frac{Q_R}{2}\right]. \qquad (6.9)$$

As we will see below, equations (6.5) and (6.9) are used as the points of departure in the search for other good estimates of the safety stock.

B. Simulation Study

A simulation program was written in SIMAN IV that simulates the repairable item inventory management system. A description of the SIMAN modeling process is provided by Reference [6]. Figure 1 shows the details of the management process and Appendix A presents the SIMAN program used in the study and a sample output. The simulation model assumes that there is a single inventory of RFI units, which are ready for issue. Demands for RFI units arrive according to a Poisson process and the mean rate is D units per quarter. A carcass accompanies the demand for an RFI unit with probability CRR and a carcass is returned with the demand with probability $1.0 - CRR$. It is assumed

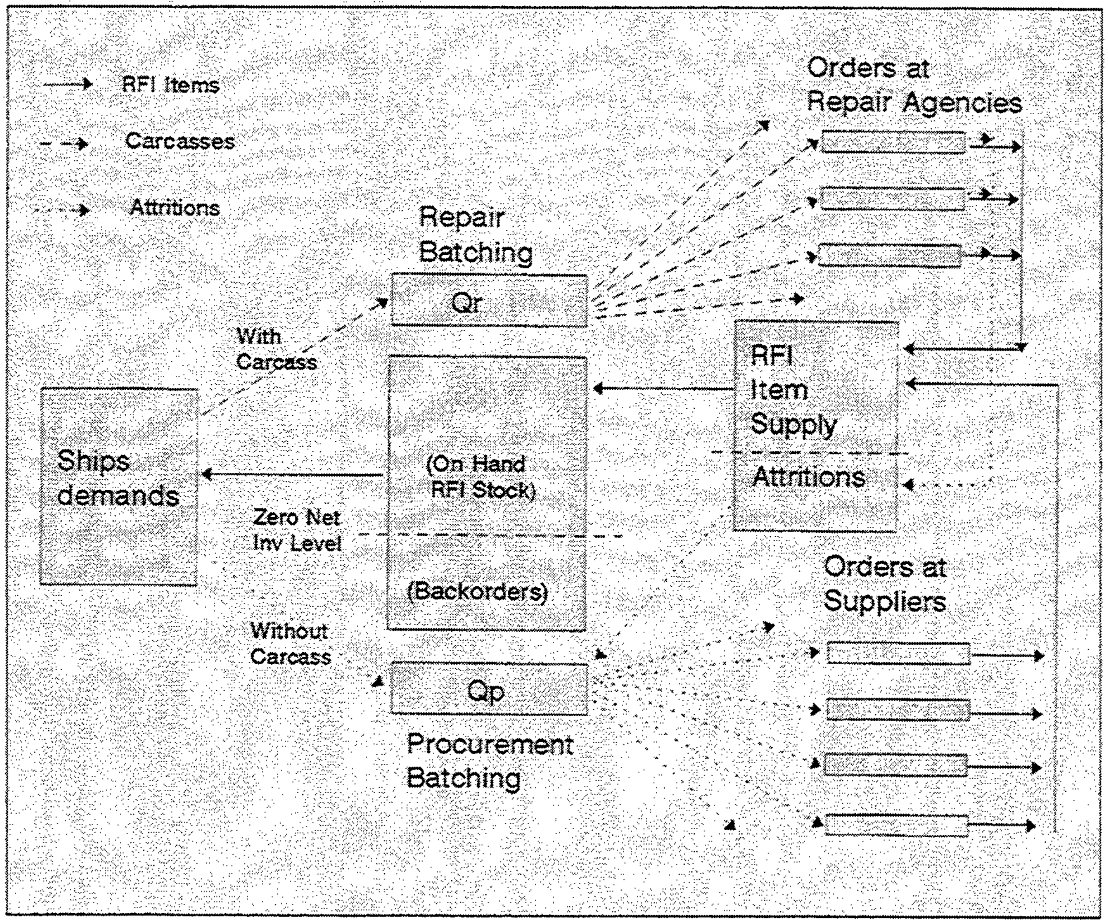

Figure 1. Flow chart of the Repairable Inventory Management Process [1].

that if a carcass is not returned with the demand then it is damaged beyond repair and can therefore be recordable as an attrition. A record of the current number of attritions is kept. When Q_P attritions have been accumulated in this record, a procurement action is immediately initiated for a batch of Q_P new units. These units arrive PCLT quarters later. Returned carcasses are accumulated until there are Q_R units. Carcasses are then sent to the depot in a batch of size Q_R. At the depot the carcasses are queued and individually evaluated to see if each can be successfully repaired. Each can be successfully repaired with probability RSR. The probability of failure is therefore $1.0 - RSR$.

Those that are failures are recorded as attritions. When carcasses are inducted for repair, the first one enters the repair process immediately. It remains in repair for RTAT quarters. After it has been in repair for REP quarters, the next carcass is inspected and either inducted or deemed to be an attrition. If it is deemed to be repairable it will be in repair for RTAT quarters. After it has been in repair for REP quarters the next carcass is inspected. As soon as a carcass has been repaired, it is returned as RFI and placed in the inventory. If $REP = 0$ then there is no delay between inspections of the individual carcasses and all repaired carcasses return to the inventory after RTAT quarters.

The study examined both the cases of $REP = 0$ and $REP > 0$. There were 70 simulation runs with $REP = 0$ and 58 with $REP > 0$. Each run was for 1000 quarters. Other parameters which were varied were SW, D, CRR, RSR, PCLT, RTAT, Q_P, Q_R and REP (for $REP > 0$). In the first part of the study the cases of pure procurement and pure repair were simulated. For pure procurement equation (6.1) was used as the estimate in a regression analysis of the simulated safety stock for this case. As can be seen in Figure 2, the results are excellent.

Pure repair needs to be simulated under both the conditions of $REP = 0$ and $REP > 0$ since these situations do exist in the Navy. For the case of $REP = 0$, equation (6.2) was used as the estimate of the safety stock. For the case of $REP > 0$, equation (6.3) was used. Figures 3 and 4 show the respective results of the simulations for pure repair.

The next analyses involved the situation where $REP = 0$ but where some units are attritions and others are successfully repaired. Several equations were evaluated for their feasibility as estimates of safety stock. Equation (6.9) was used

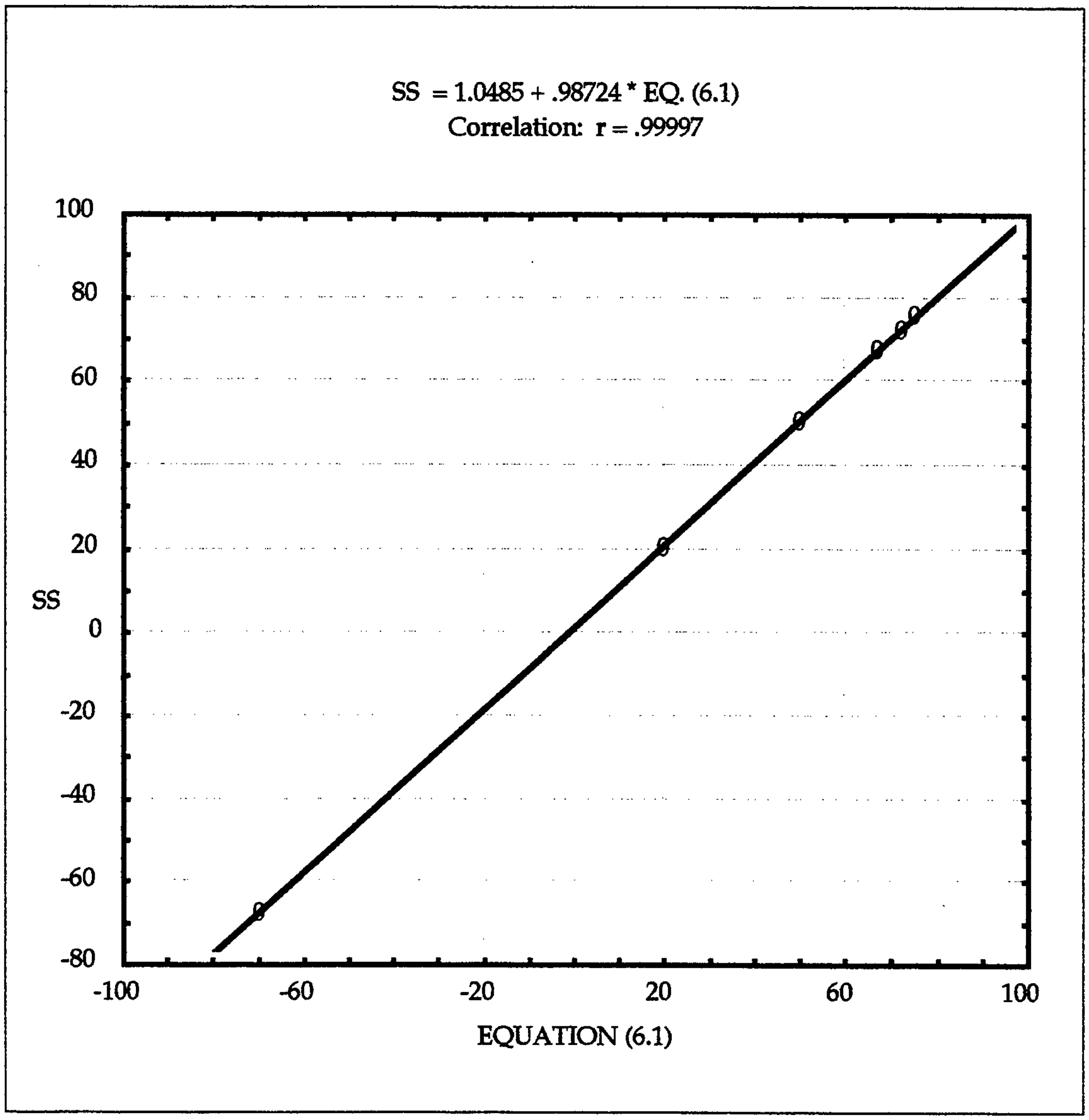

Figure 2. Pure Procurement.

as a starting point for the search for good equations that would fit the safety stock data well. Many variants were tried. However, only those which were as good or better than equation (6.5) were considered as potential alternative estimates of safety stock. The best fit was obtained from equation (6.10).

$$SS = SW - PPV - Q_P e^{-CRR*RSR} - Q_R e^{-(1.0-CRR*RSR)} . \qquad (6.10)$$

The other equations which were tested and performed nearly as well

were combinations/variations of equations (6.5) and (6.10). For example,

$$SS = SW - PPV - Q_P(1.0 - CRR * RSR) - Q_R RSR, \qquad (6.11)$$

and

$$SS = SW - PPV - Q_P e^{-CRR * RSR} - Q_R CRR * RSR. \qquad (6.12)$$

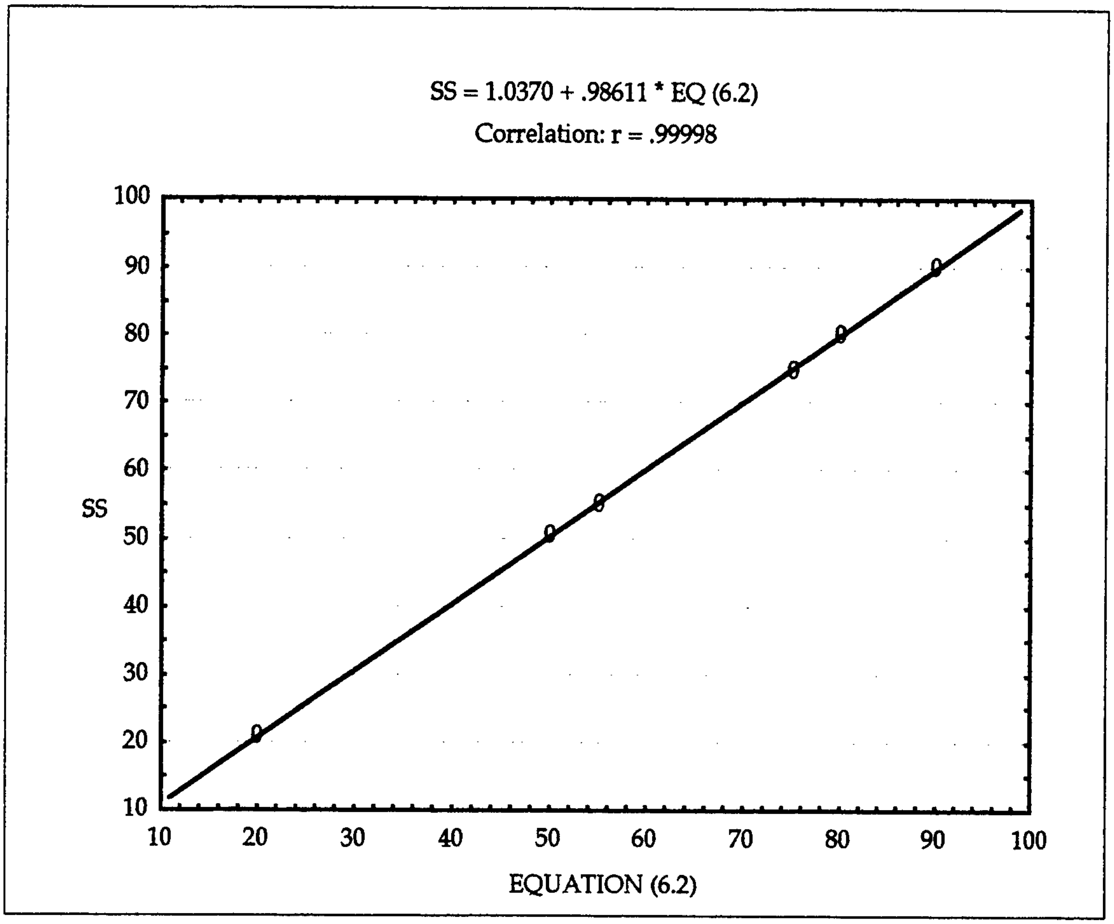

Figure 3. Pure Repair with $REP = 0$.

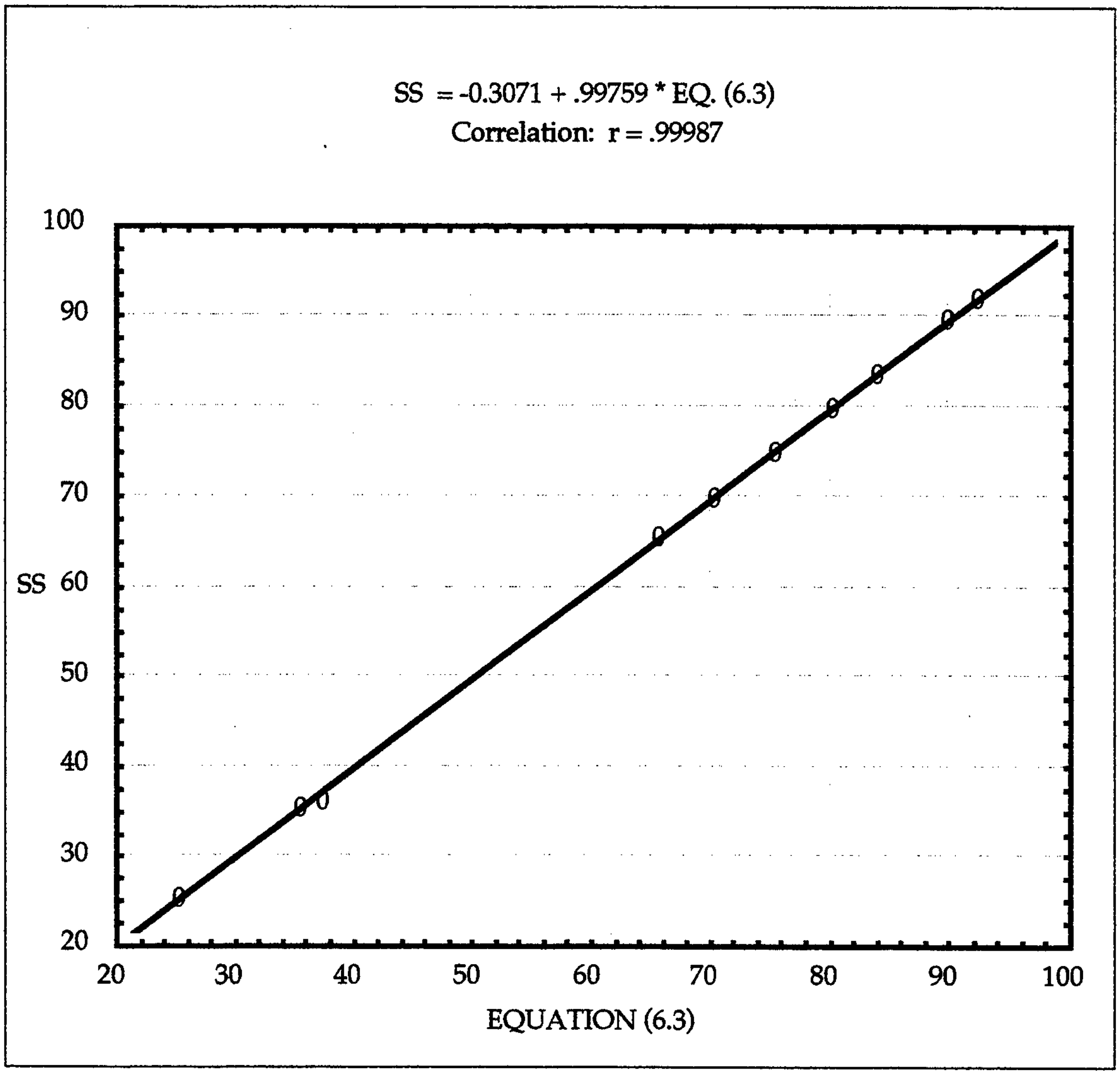

Figure 4. Pure Repair with $REP > 0$.

Figure 5 shows the fit of equation (6.5) to all of the simulated safety stock data; that is, all data including pure procurement and pure repair. Figure 6 shows the fit of equation (6.10). The regression equation for the fit of equation (6.11) was

$$SS = -0.6851 + .98932 * EQ.(6.11)$$

which has an $r = 0.99835$.

The regression equation for the fit of equation (6.12) was

$$SS = -1.790 + 1.0082 * EQ.(6.12)$$

and has $r = 0.99811$.

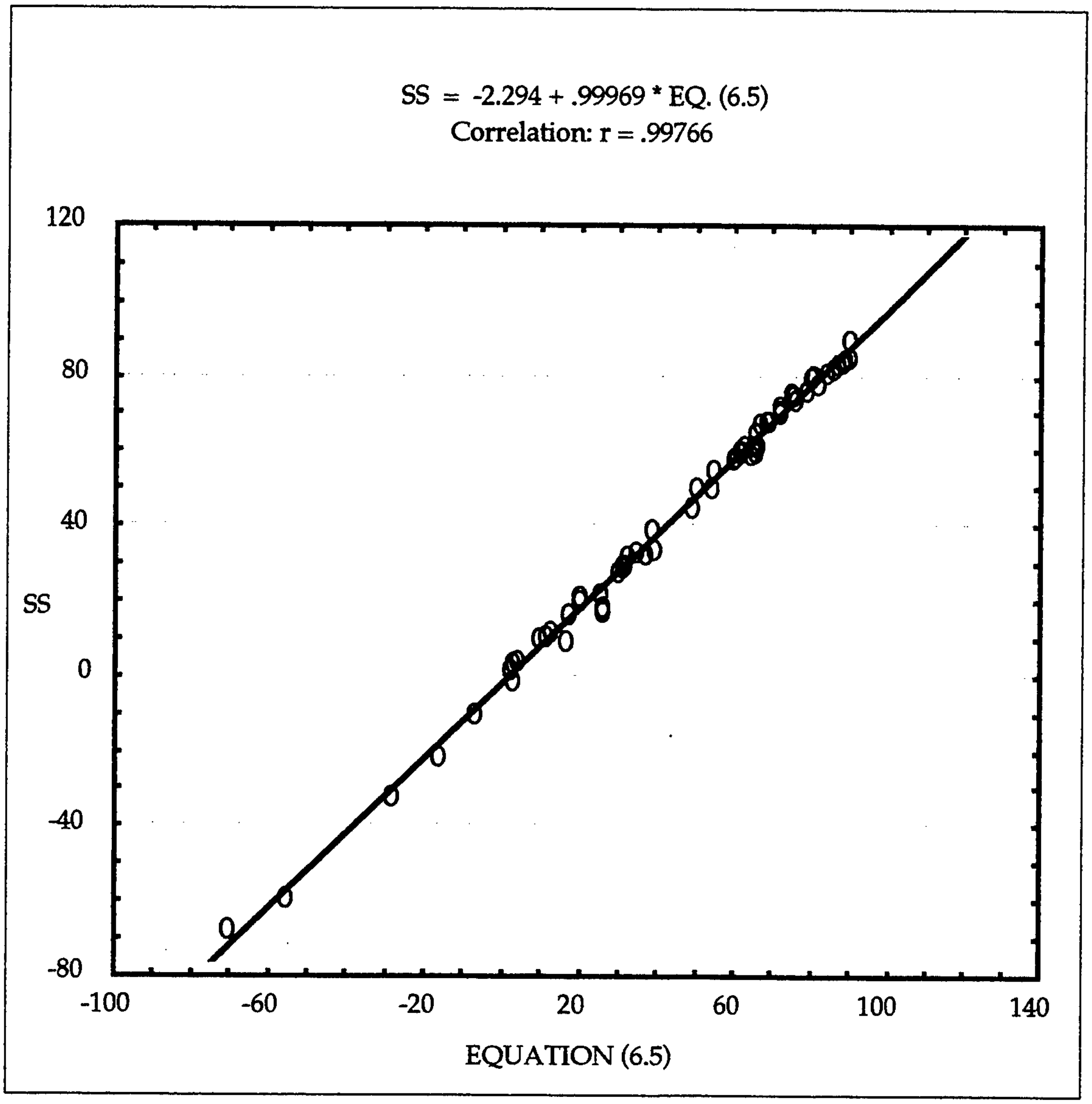

Figure 5. Equation (6.5) applied to all data for REP=0.

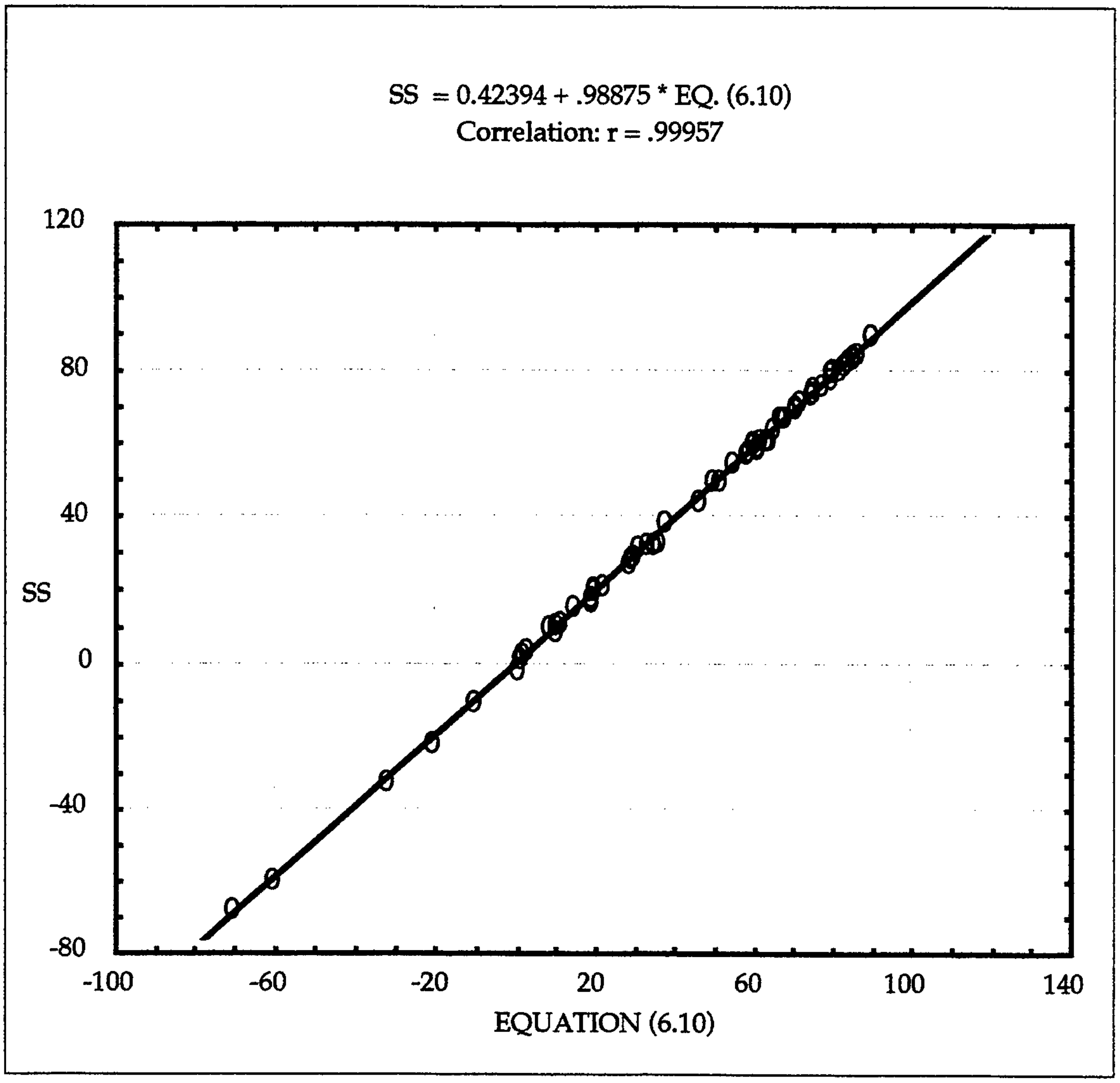

Figure 6. Equation (6.10) applied to all data for REP=0.

Next we consider the case where $REP > 0$ and there are both attritions and repairs. The base case here was given by equation (6.9). The regression results are presented in Figure 7. The equivalent of equation (6.10) for the case where $REP > 0$ is:

$$SS = SW - ZB - Q_P e^{-CRR*RSR} - \frac{Q_R}{2} e^{-(1.0-CRR*RSR)} . \qquad (6.13)$$

Figure 8 shows the regression results when equation (6.13) is used

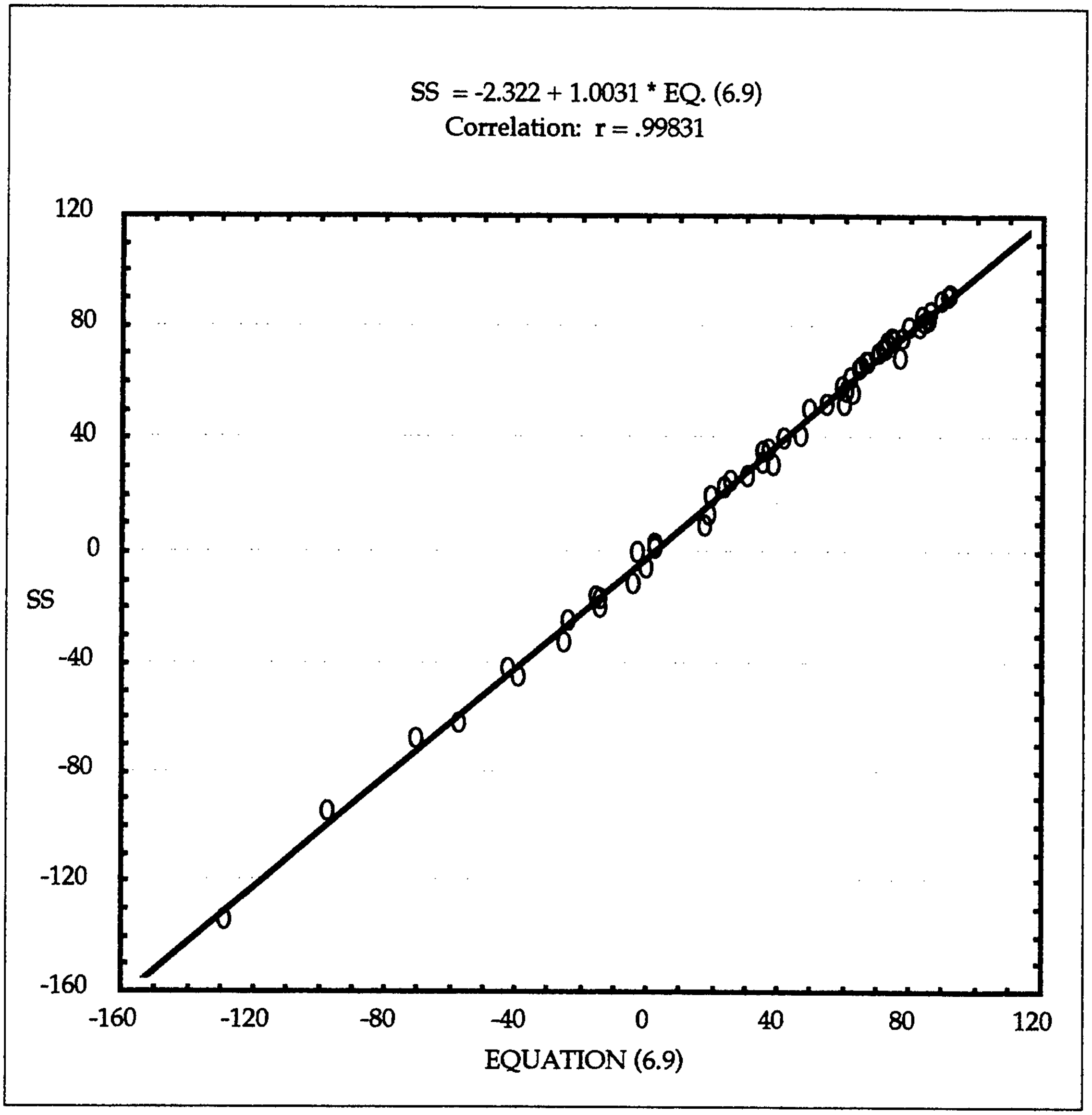

Figure 7. Equation (6.9) for all data for REP>0.

Other equations, which were examined, include

$$SS = SW - ZB - Q_P(1 - CRR * RSR) - \frac{Q_R}{2} RSR, \tag{6.14}$$

and

$$SS = SW - ZB - Q_P e^{-CRR * RSR} - \frac{Q_R}{2} CRR * RSR. \tag{6.15}$$

As can be seen, these are similar to equations (6.9) and (6.13) above. Equation (6.14) regression results were

$$SS = -1.312 + 0.99855 * EQ.(6.14),$$

with $r = 0.99836$. The regression results for equation (6.15) are shown below in Figure 9. This equation gives results which are very close to those of equation (6.13).

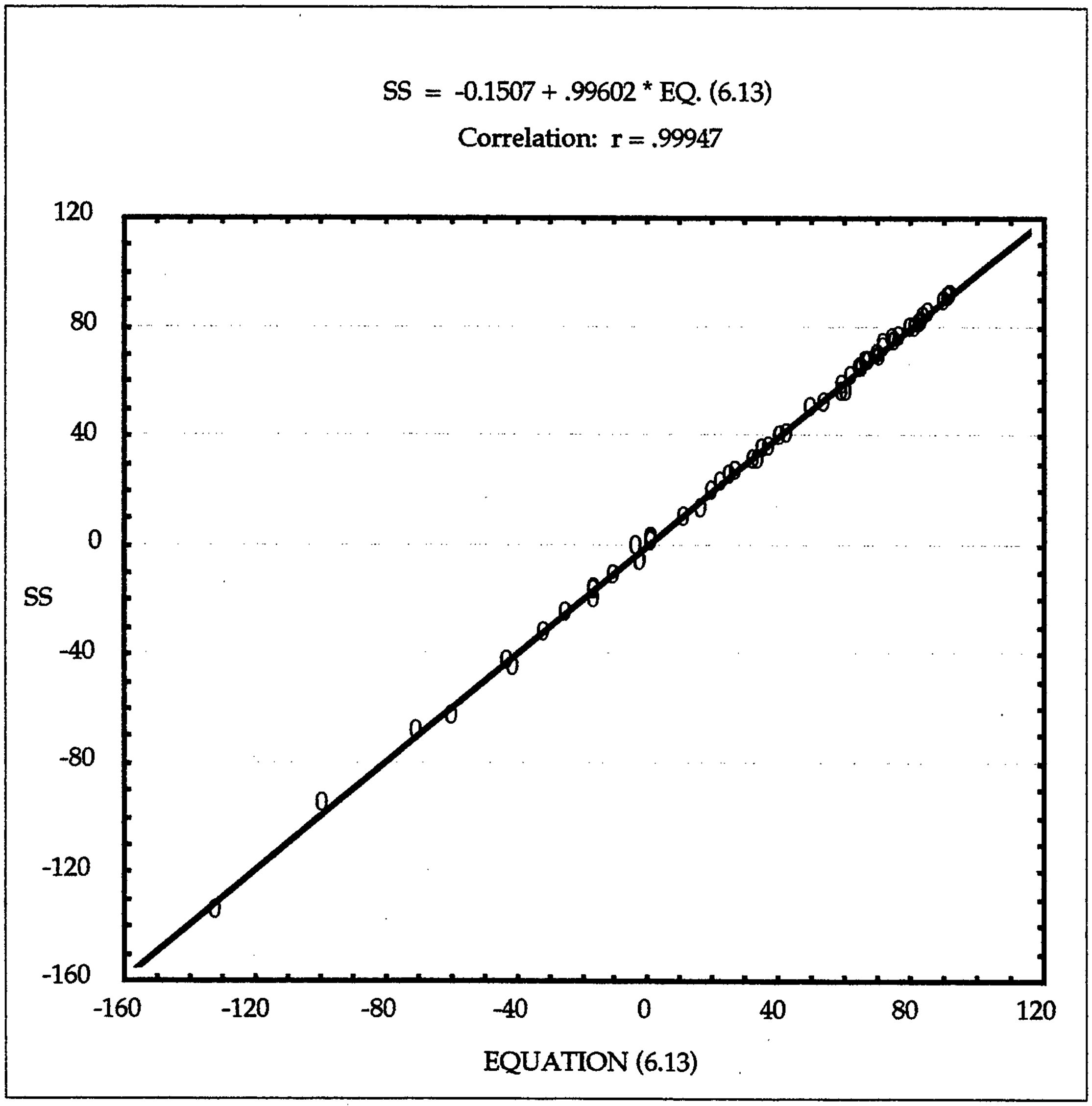

Figure 8. Equation (6.13) for all data for REP>0.

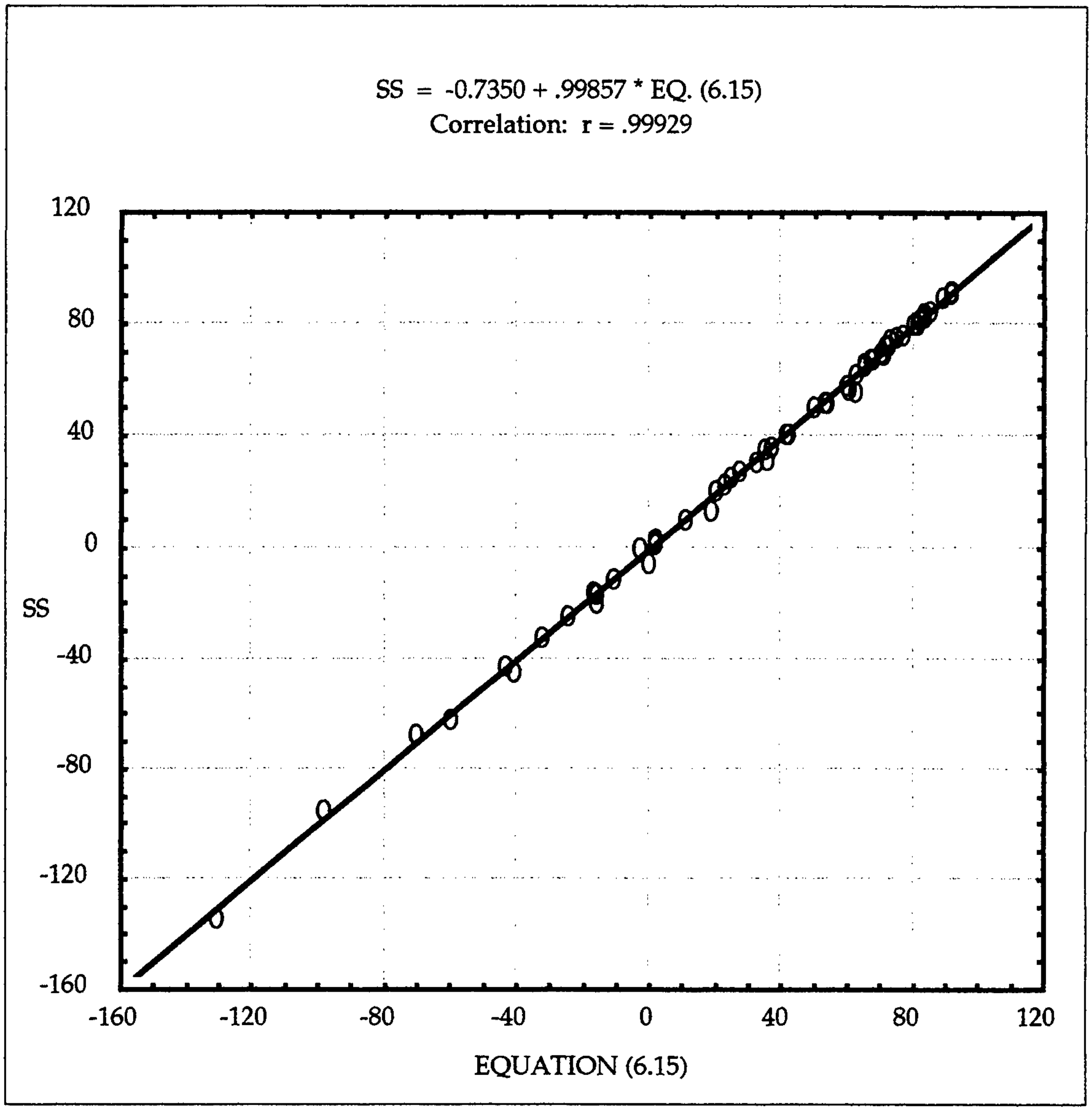

Figure 9. Equation (6.15) for all data for REP>0.

Another equation provided an interesting different way of making the convex combination of Q_P and Q_R. It was

$$SS = SW - ZB - \left[\frac{Q_P(1 - CRR * RSR)PCLT + \frac{Q_R}{2} CRR * RSR(RTAT + \frac{Q_R - 1}{2} REP)}{(1 - CRR * RSR)PCLT + CRR * RSR(RTAT + \frac{Q_R - 1}{2} REP)} \right]. \tag{6.16}$$

This represents a weighting which comes from the two parts of the modified form of L_2 for the case of $REP > 0$.

$$L_2 = (1 - CRR * RSR)PCLT + CRR * RSR(RTAT + \frac{Q_R - 1}{2} REP). \quad (6.17)$$

Figure 10 shows that the fit is not as good as the base equation (6.9), however. The equivalent form of equation (6.16) for the case of $REP = 0$ also gave a poorer fit ($r = 0.99728$) than the base equation (6.5) for that case.

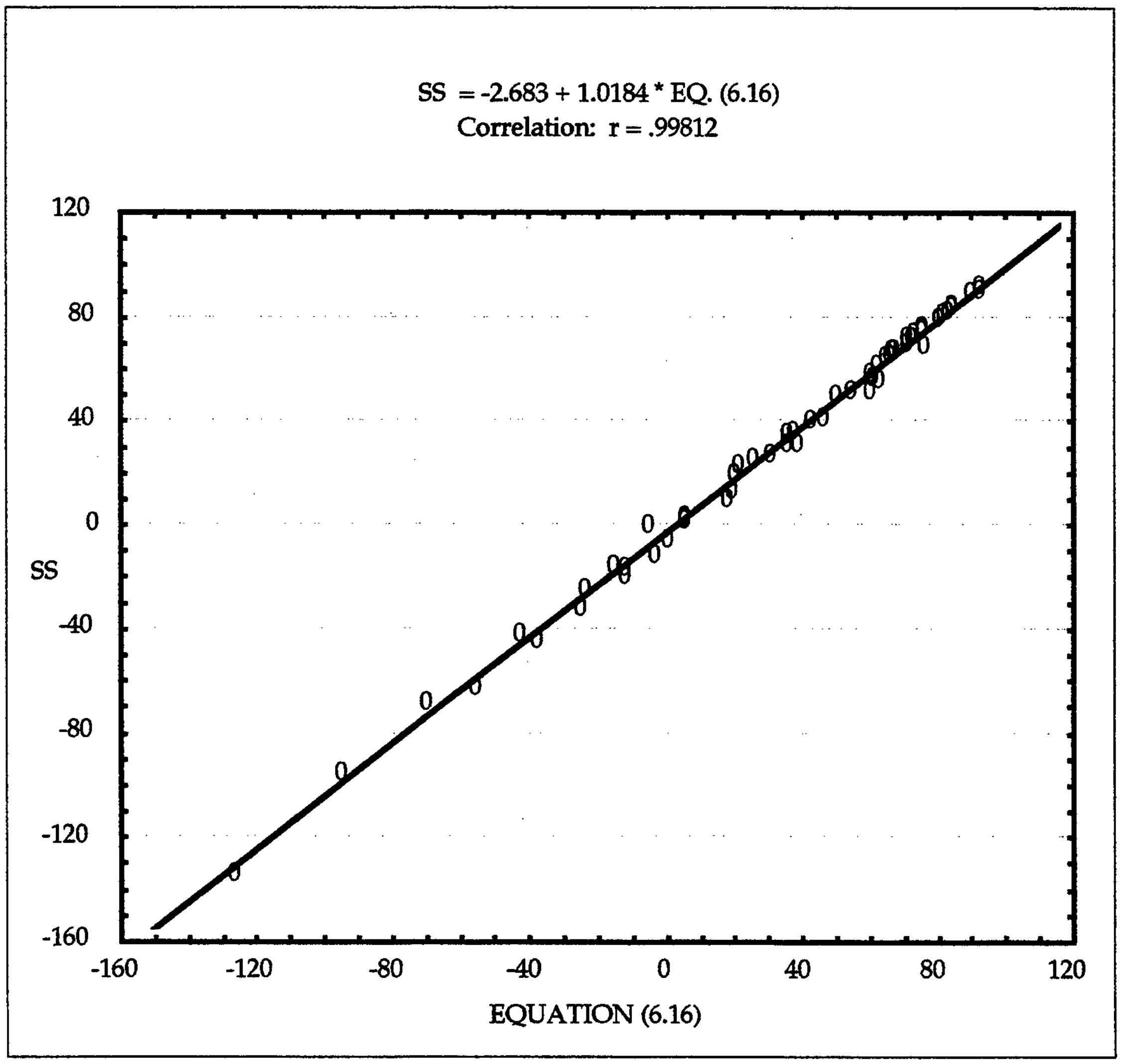

Figure 10. Equation (6.16) for all data for REP>0.

C. Conclusions

To place the simulation regression results in the proper perspective it is useful to note that Professor Keebom Kang. a simulation expert at the Naval Postgraduate School, expressed amazement at how well all of the equations presented above fit the simulated data. He noted that an $r > 0.8$ is considered very good for such approximating equations in fitting simulation results for complex processes. Thus, the user can decide on which of these equations suits his intuition and be assured that he has a good estimate for the safety stock for a given set of parameters.

It is important to mention here that the equations for safety stock presented in this chapter are applicable only for the conditions of Baker's model; namely, that demand during the aggregate lead time is Poisson distributed. In the simulation model described above the demand rate is Poisson and the aggregate lead time is a constant in PCLT, RTAT and REP. The aggregate lead time L_2 may seem, however, to really not be constant because of the CRR and RSR terms as equations (6.7) and (6.17) show. Fortunately, Baker [1] noted that since the individual parts are Poisson distributed (i.e., demand during PCLT and demand during RTAT), the CRR and RSR terms are weights associated with the summing of two Poisson distributed random variables so that demand during the aggregate lead time is also Poisson.

The conditions associated with the derivations in Chapters 4 and 5 for the assumption that the demand during aggregate lead time is Normally distributed can be expected to lead to different equations for the safety stock. This will be the subject of a future study.

CHAPTER 7 - APPROXIMATE FORMULAS FOR LEAST COST Q_P AND Q_R

A. Introduction

This chapter presents the results of a study to determine approximate equations for the values of Q_P and Q_R which minimize the average annual total variable costs for managing an inventory of an item. The costs include the costs to buy new units (procurement), the costs to repair carcasses, the costs to hold items in inventory, and the costs of backorders. The purpose in doing this was to reduce the amount of computer time required to determine the maximum inventory position values which will minimize the aggregate mean supply response time (MSRT) for a group of items belonging to a weapon system.

At present the UICP model [5] uses the following "optimal values" for Q_P and Q_R.

$$Q_P = \sqrt{\frac{8A(D-G)}{IC}} \qquad Q_R\sqrt{\frac{8A_2 Min(D,G)}{IC_2}}; \qquad (7.1)$$

where,

D = Quarterly Demand Rate;

A = Procurement Contract Costs;

A_2 = Repair Contract Costs;

C = Unit Purchase Cost;

C_2 = Unit Repair Cost;

I = Holding Cost Rate = 0.21;

$G = CRR * RSR * D$;

CRR = Probability of a Carcass Being Returned by a Customer;

RSR = Probability of a Carcass being Repaired.

These equations were derived from conjectured UICP average annual total variable costs equations, one for managing newly procured units and one for managing repairing of carcasses and the inventory of successful repaired units.

The new repairable model has a cost equation which combines the two parts conjectured by the UICP modelers since the repaired units and newly procured units are kept in the same inventory as equally valid ready-for-issue (RFI) units. This equation can be written as

$$TVC = \frac{4A(D-G)}{Q_P} + \frac{4A_2 CRR * D}{Q_R} + IC_3 EOH + \lambda B(SW), \qquad (7.2)$$

where EOH represents the expected on-hand inventory. Since the expected net inventory, E(NI), is the difference between the expected on-hand and the expected number of backorders, B(SW), we can write $EOH = E(NI) + B(SW)$, which, upon substitution of the E(NI) formula of Baker [1], gives

$$EOH = SW - ZB - \frac{(Q_P + Q_R - 2)}{2} + B(SW).$$

Here,

SW = Maximum Inventory Position,

ZB = Baker's Extension for PPV (given by equation (6.8)).

In equation (7.2) the average annual number of repair contracts is given as $\frac{4 * CRR * D}{Q_R}$. The number of accumulated carcasses per year is the numerator since CRR represents the probability of a carcass being returned per replacement unit demanded. The denominator is Q_R since that is the number of carcasses inducted as a batch (Q_R does not represent the number of carcasses repaired).

The expected annual backorder cost term in equation (7.2) is given by $\lambda B(SW)$. Formulas for computing $B(SW)$ were derived in Chapters 3 and 5 of this report. The shortage cost per year per unit is represented by λ. In the UICP model the shortage cost is per requisition and there is also an essentiality term added into the backorder costs. Thus, if such a form is of interest then we can write the equivalence as $\lambda = \frac{E * \lambda_{UICP}}{F}$ where E represents the number for essentiality (currently 0.5 for Mechanicsburg) and F represents the average requisition size. In the new model $F = 1.0$ is assumed (a review of repairable item data shows this to be approximately true).

Finally, the holding cost rate in equation (7.2) includes the average unit cost, called C_3, which is a convex combination of the unit purchase cost and the unit repair cost.

$$C_3 = \left[1 - \frac{G}{D}\right]C + \frac{G}{D}C_2 . \qquad (7.3)$$

C. Computer Runs

A computer program was written in FORTRAN 77 to determine the optimal values for Q_P and Q_R which will minimize equation (7.2) for a wide range of parameter values. These included D, CRR, RSR, PCLT, RTAT, REP, C, C_2, A, A_2, and λ. The program was designed to run for a fixed set of these parameters over a range of SW values from 0 to 150 or 200 in increments of 10. Sixty-five such sets were obtained.

The approach taken in the program is to incrementally increase Q_R starting at 1 and then incrementally increase Q_P to find the optimal Q_P, Q_P^*, for

each Q_R value. As long as $TVC(Q_P^*, Q_R)$ is decreasing the value of Q_R is increased. This search process continues until the next value of Q_R shows an increase in $TVC(Q_P^*, Q_R)$. Optimal Q_R is then the previous value of Q_R.

The result of these computer runs were then recorded in a STATISTICA 4.1 [8] file. Files were developed for a range of values of the product $CRR * RSR$ and each of the other parameters.

Only one parameter was allowed to vary at a time. Plots of the Q_P and Q_R values as a function of a parameter were examined to see what function of that parameter would provide a good regression fit. These were often quite complex nonlinear functions. When the functions had all been determined, they were assumed to be combinable into a sum of separate functions (cross products and other combinations were examined prior to deciding on this approach) which was then applied to the complete set of data to see how good the aggregate fit was. The results were surprising good.

In the process of conducting the analyses it was discovered that when $SW - ZB \leq 0.0$ that the optimal values of Q_P and Q_R remained unchanged as SW was increased. Figure 11 shows the typical behavior of optimal Q_P and Q_R as a function of $SW - ZB$.

Mathematical analyses of TVC to see what formulas could be derived for unit changes in Q_P and Q_R were conducted. The first analysis was to find the relation for determining optimal Q_P. Using the technique of finite differences, we seek the largest value of Q_P for which

$$\Delta TVC(Q_P) = TVC(Q_P) - TVC(Q_P - 1) \leq 0.$$

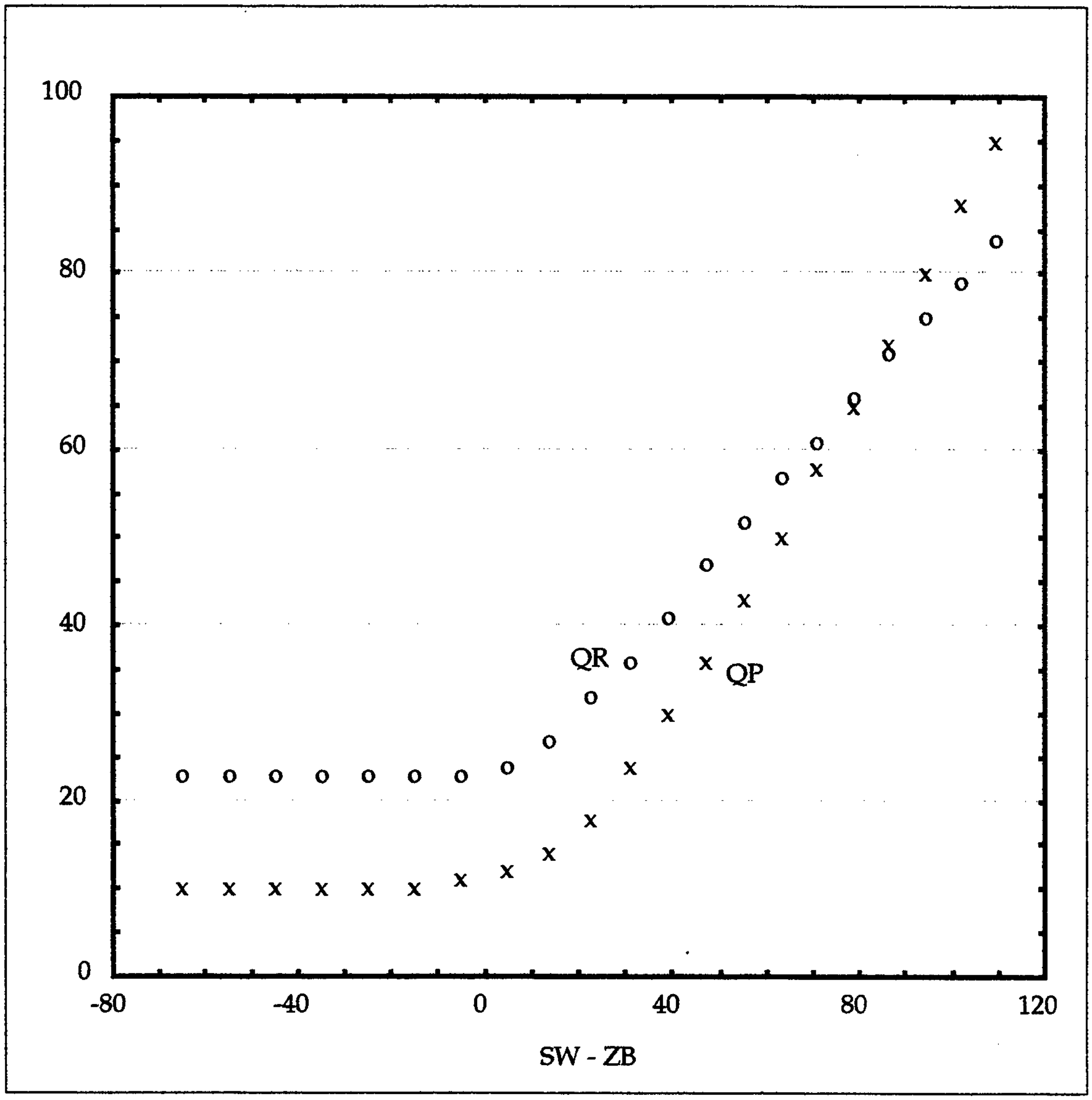

Figure 11. Optimal Q_P and Q_R as a function of $SW - ZB$.

$$
\begin{aligned}
\Delta TVC(Q_P) = {} & \frac{4A(D-G)}{Q_P} + \frac{4A_2 CRR * D}{Q_R} \\
& + IC_3 \left[\begin{array}{l} SW - D(1 - CRR * RSR)PCLT \\ -D * CRR * RSR\left(RTAT + \frac{Q_R - 1}{2} REP \right) \\ -\frac{Q_P + Q_R - 2}{2} + B(SW, Q_P) \end{array} \right] \\
& + \lambda B(SW, Q_P) - \frac{4A(D-G)}{Q_P - 1} - \frac{4A_2 CRR * D}{Q_R} \\
& - IC_3 \left[\begin{array}{l} SW - D(1 - CRR * RSR)PCLT \\ -D * CRR * RSR\left(RTAT + \frac{Q_R - 1}{2} REP \right) \\ -\frac{(Q_P - 1) + Q_R - 2}{2} + B(SW, Q_P - 1) \end{array} \right] \\
& - \lambda B(SW, Q_P - 1).
\end{aligned}
\tag{7.4}
$$

After collecting terms and canceling common terms in equation (7.4) we have

$$
\Delta TVC(Q_P) = -\frac{4A(D-G)}{Q_P(Q_P - 1)} - \frac{IC_3}{2} + (IC_3 + \lambda)\Delta B(SW, Q_P) \le 0, \tag{7.5}
$$

where

$$
\Delta B(SW, Q_P) = B(SW, Q_P) - B(SW, Q_P - 1).
$$

From inequality (7.5) we can write the optimality condition as

$$
Q_P(Q_P - 1) \le \frac{8A(D-G)}{2(IC_3 + \lambda)\Delta B(SW, Q_P) - IC_3}. \tag{7.6}
$$

No attempt was made to use the formulas for $B(SW)$ from Chapter 3 because of their complexity. Instead, a detailed investigation of the computer results was conducted. It showed that for $SW \le ZB$ that $\Delta B(SW, Q_P) = 0.5$ in the neighborhood of optimal Q_P. Therefore, inequality (7.6) reduces to

$$Q_P(Q_P-1) \le \frac{8A(D-G)}{\lambda}. \tag{7.7}$$

Thus, we want the largest value of Q_P for which inequality (7.7) is satisfied. A check of the computer results also showed that the optimal Q_P from the computer search procedure was that value which satisfied inequality (7.7) when $SW \le ZB$.

To find an inequality for determining optimal Q_R the approach used was similar to that for Q_P. Now we want the largest value of Q_R for which

$$\Delta TVC(Q_R) = TVC(Q_R) - TVC(Q_R - 1) \le 0 .$$

In this case

$$\Delta TVC(Q_R) = -\frac{4A_2CRR*D}{Q_R(Q_R-1)} - \frac{IC_3}{2}(1+G*REP) + (IC_3+\lambda)\Delta B(SW,Q_R). \tag{7.8}$$

The computer results in the neighborhood of optimal Q_R were not a constant like was the case for Q_P. Values were recorded in a STATISTICA 4.1 file [8] for a variety of parameters and their values. Regression analyses discovered that

$$\Delta B(SW,Q_R) = \frac{1}{2}(1+G*REP). \tag{7.9}$$

Figure 12 shows the impressive fit of the data to equation (7.9). As a consequence, the optimality condition for Q_R when $SW \le ZB$ is that it is the largest value of Q_R for which

$$Q_R(Q_R-1) \le \frac{8A_2CRR*D}{\lambda(1+G*REP)}. \tag{7.10}$$

Inequality (7.10) was confirmed by the computer program results.

When $SW \geq ZB$ it is easy to see that optimal Q_P and Q_R are approximately linear in the difference $SW - ZB$. However, an attempt to develop some relationship between $\Delta B(SW, Q_P)$ or $\Delta B(SW, Q_R)$ and optimal Q_P or Q_R was unsuccessful. The observed values of $\Delta B(SW, Q_P)$ and $\Delta B(SW, Q_R)$ become erratic and quite small in value as SW increases. Other terms of the

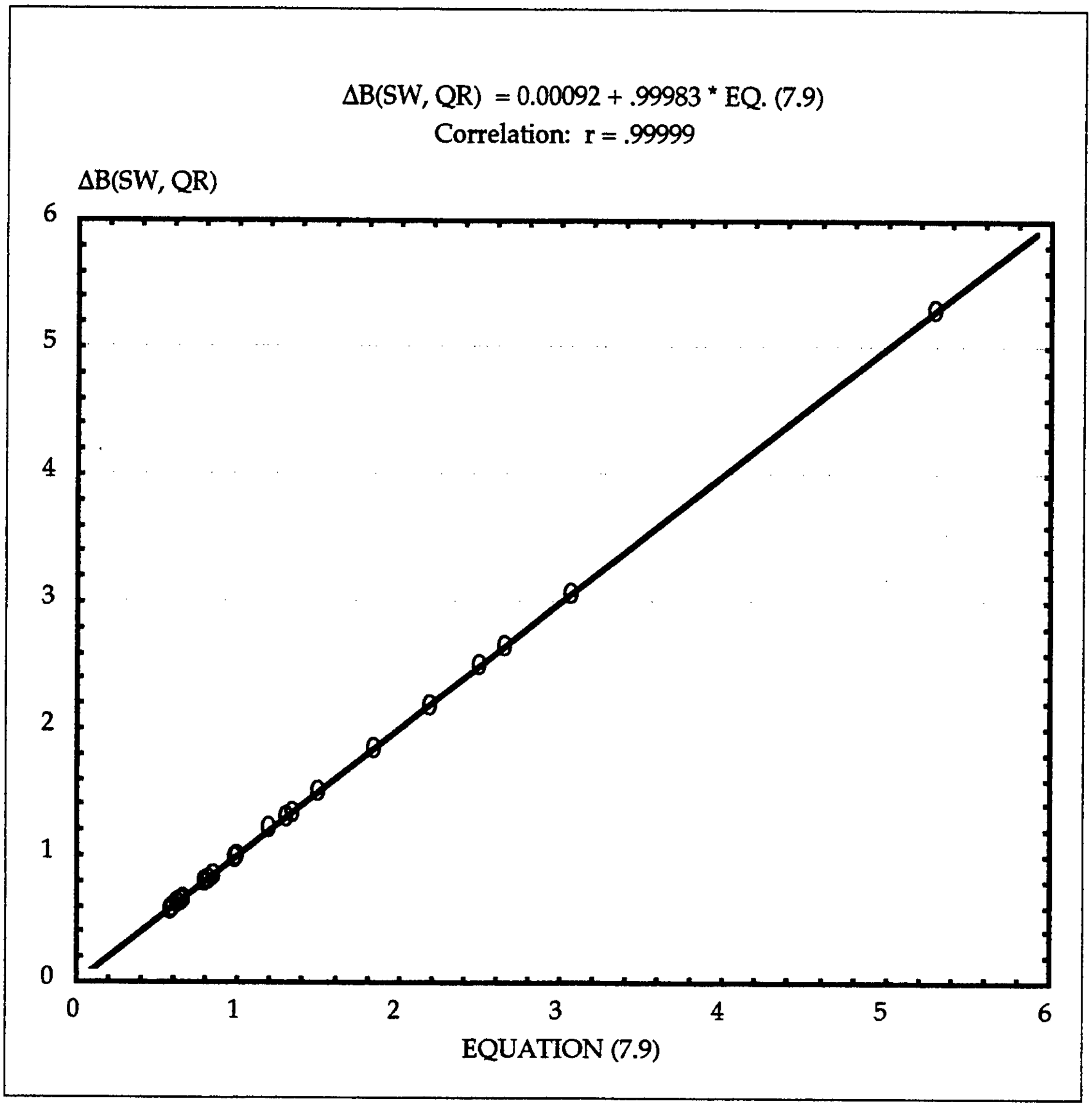

Figure 12. The Regression Fit of Equation (7.9) to $\Delta B(SW, Q_R)$ Data.

TVC equation also dominate the determination of optimal Q_P and Q_R as SW increases. As a consequence, an investigation of the relationships between optimal Q_P and Q_R and each of the parameters (except I) of TVC was begun.

As mentioned earlier, the process consisted of varying the value of a parameter while all of the others remained fixed. From plots of the values of optimal Q_P or Q_R for a range of a parameter's value functions of the parameters were suggested and fits attempted. Then improvements were made and tested. This iterative process eventually closed in on the best fitting function.

Figure 11 shows optimal Q_P and Q_R to be a linear function of $SW - ZB$. We know from the equation for ZB given by equation (6.8) that it contains D, CRR, RSR, PCLT, RTAT, REP, and Q_R. However, there are more functions of these and other parameters which affect the slopes and intercepts of those straight lines when $SW \geq ZB$.

To illustrate the process, we will use the analysis of C_3's effect on optimal Q_P. We first define

$$DELQP \equiv Q_P^* - Q_P^*(SW = 0),$$

where $Q_P^*(SW = 0)$ satisfies equation (7.7) when $SW = 0$.

Figure 13 shows curves of DELQP as a function of $SW - ZB$ for the range of C_3 values from \$720 (the lower curve) to \$10,800 (the upper curve). Since we already know how to compute $Q_P^*(SW = 0)$ we focus on $SW > ZB$.

Figure 13 shows the plots of DELQP are fairly linear in $SW - ZB$ but the slopes and intercepts change as C_3 increases. Figures 14 and 15 show the slopes and intercepts for the extremes of the C_3 values resulting from regression fits.

For each of the curves of Figure 13 the slopes and intercepts were determined as was done in Figures 14 and 15. These slope and intercept values were then

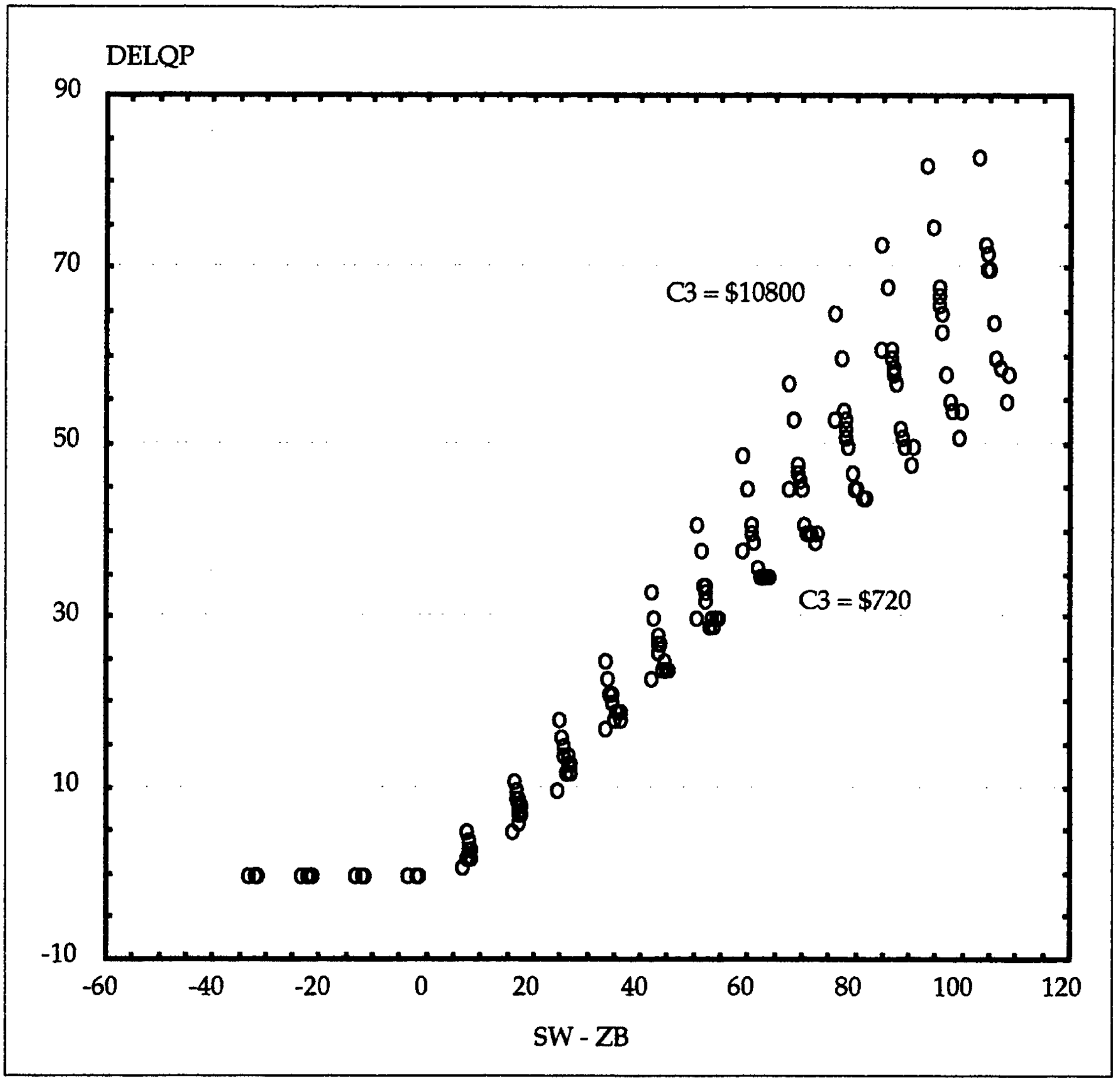

Figure 13. Optimal Values of DELQP for a Range of C_3 Values.

plotted as a function of C_3 as shown in Figures 16 and 17. Using a lot of guess work. various functions were conjectured and tried until something reasonably close to each of the curves in Figures 16 and 17 was found. In Figure 16 QPSLP stands for the slope of the Q_P curve for a certain C_3 value. In Figure 17 QPINT stands for the intercept of the Q_P curve at $SW = ZB$.

For QPSLP. the function $\left(\frac{C_3}{1000}\right)^{0.22}$ was found to be close. A regression fit with QPSLP suggested that a coefficient of 0.554 and an intercept constant of -0.031 was needed. Thus, the equation which will be used as an estimate for QPSLP is

$$QPSLP(C_3) = -0.031 + 0.554\left(\frac{C_3}{1000}\right)^{0.22}. \tag{7.11}$$

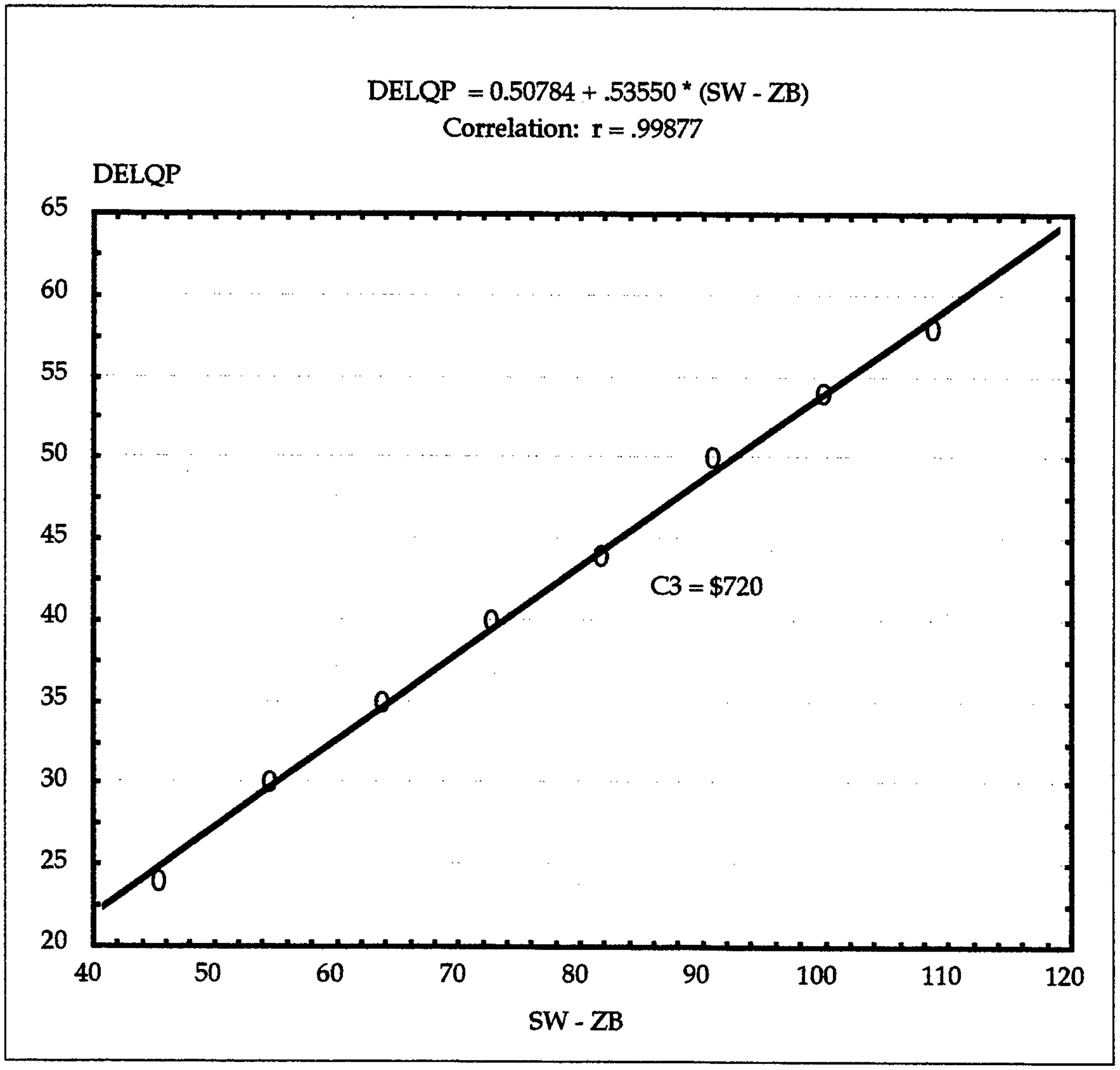

Figure 14. Regression Fit of DELQP with $SW - ZB$ for $C_3 = \$720$ when $SW > ZB$.

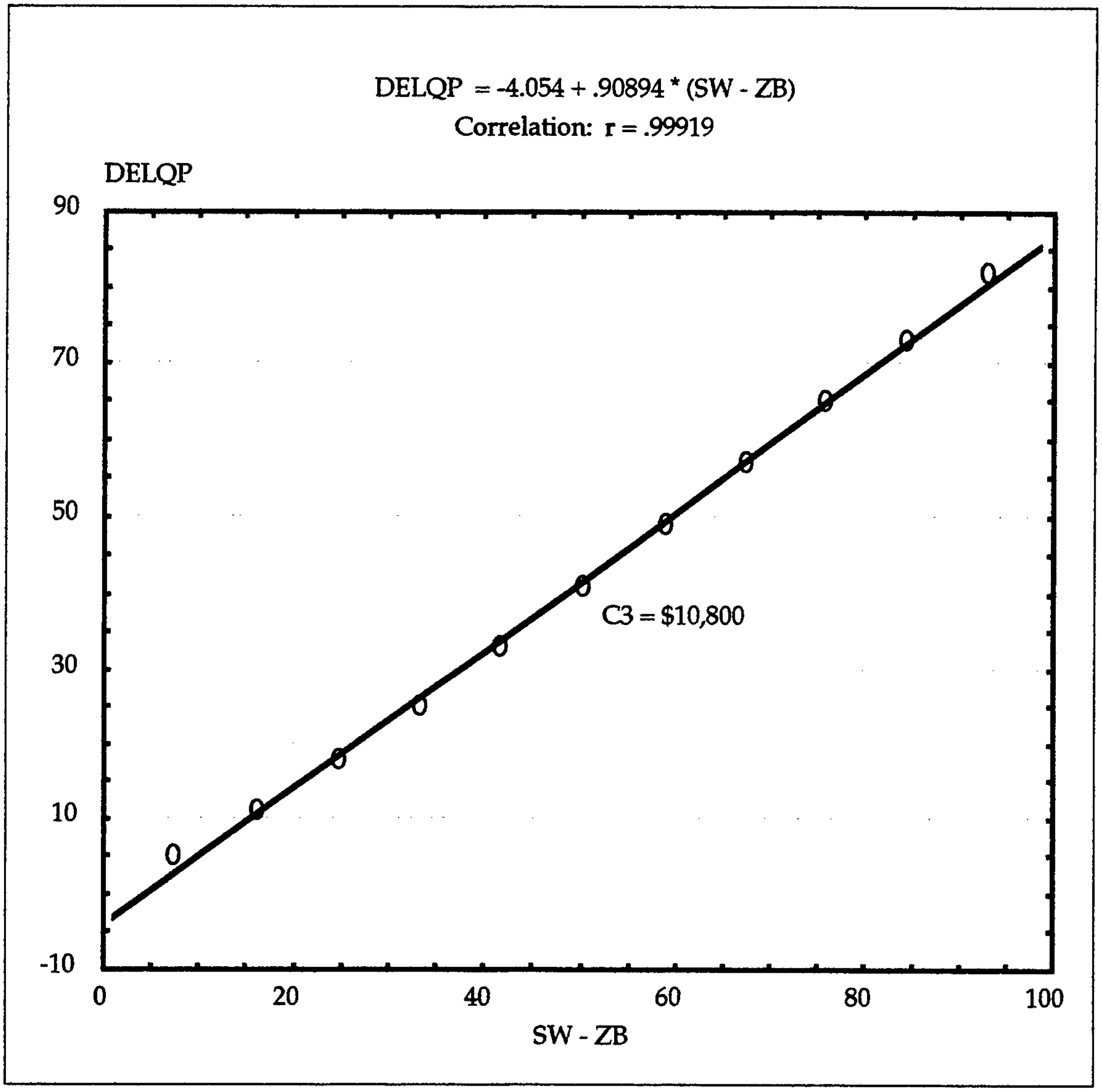

Figure 15. Regression Fit of DELQP with $SW-ZB$ for $C_3 = \$10,800$ when $SW > ZB$.

Figure 18 shows the regression fit between QPSLP and equation (7.11). The value of the correlation coefficient r is 0.98092 shows the fit is good. Figure 19 shows how the two of them compare when they are plotted as functions of C_3.

The intercept part, QPINT, was more difficult to develop a function for. The equation which was finally chosen is

$$QPINT(C_3) = -5.86 + 0.187\left(\frac{C_3}{1000} - 7.0\right)^2. \tag{7.12}$$

The regression fit gave $r = 0.94565$ which is not as good as equation (7.11) for QPSLP. The comparative plots of equation (7.12) and QPINT as a function of C_3 are shown in Figure 20. Clearly, equation (7.12) provides only approximate results.

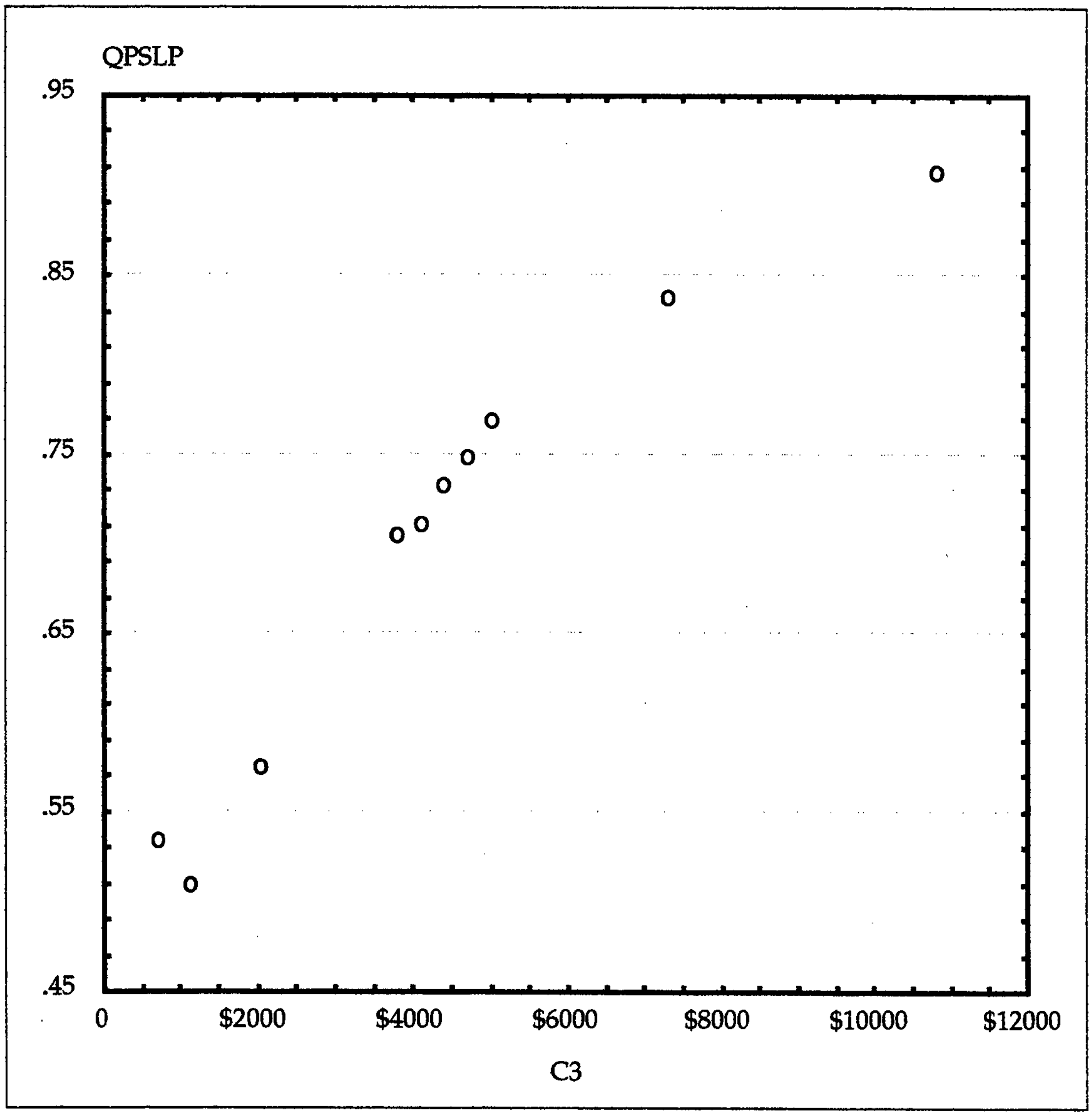

Figure 16. Slopes of C_3 Curves for Optimal Q_P when $SW > ZB$.

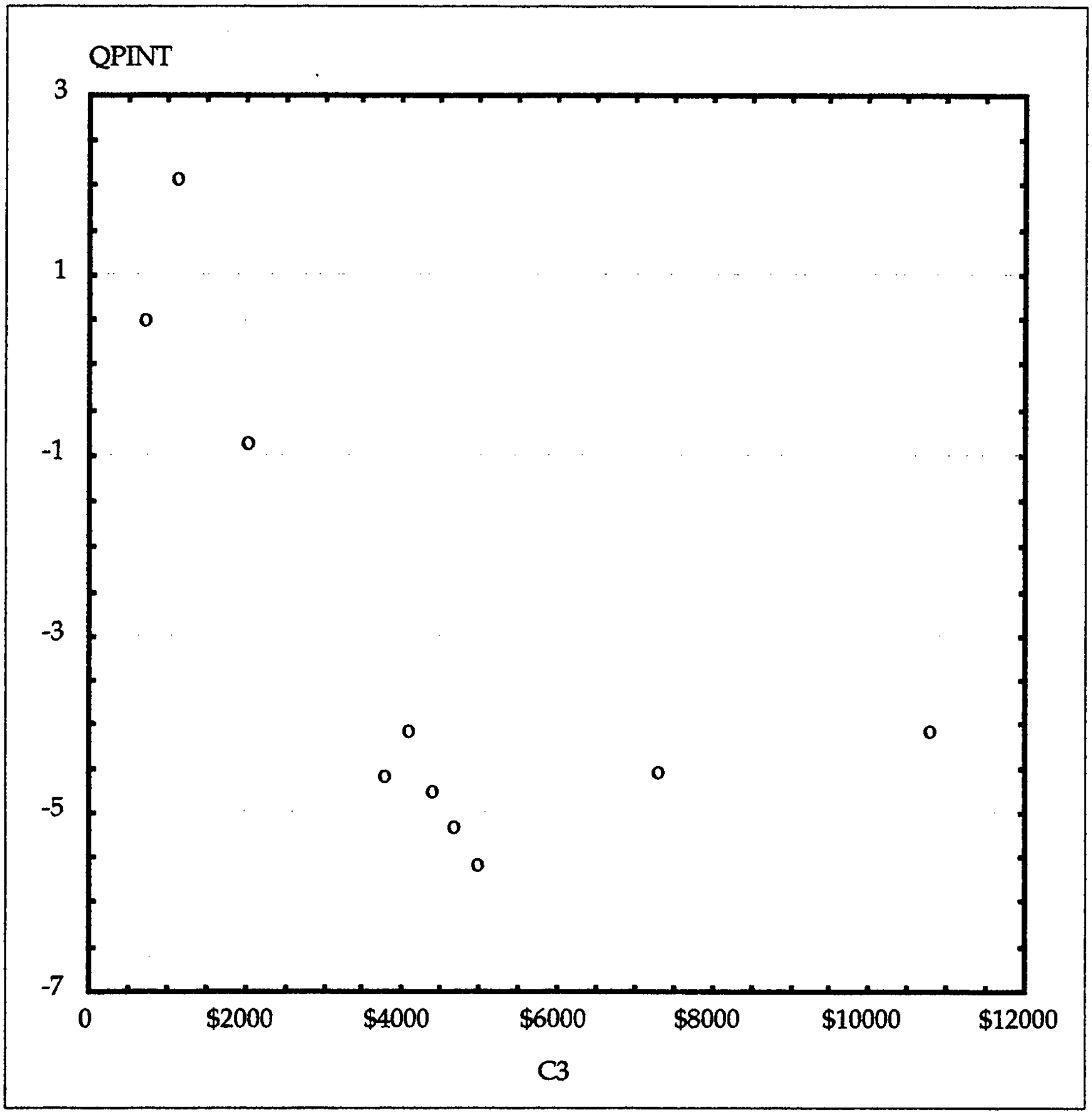

Figure 17. Intercepts of C_3 Curves for Optimal Q_P when $SW > ZB$.

The other approximate formulas for the slopes and intercepts for Q_P for the other parameters are given in Table 1. Table 2 provides comparable information for Q_R.

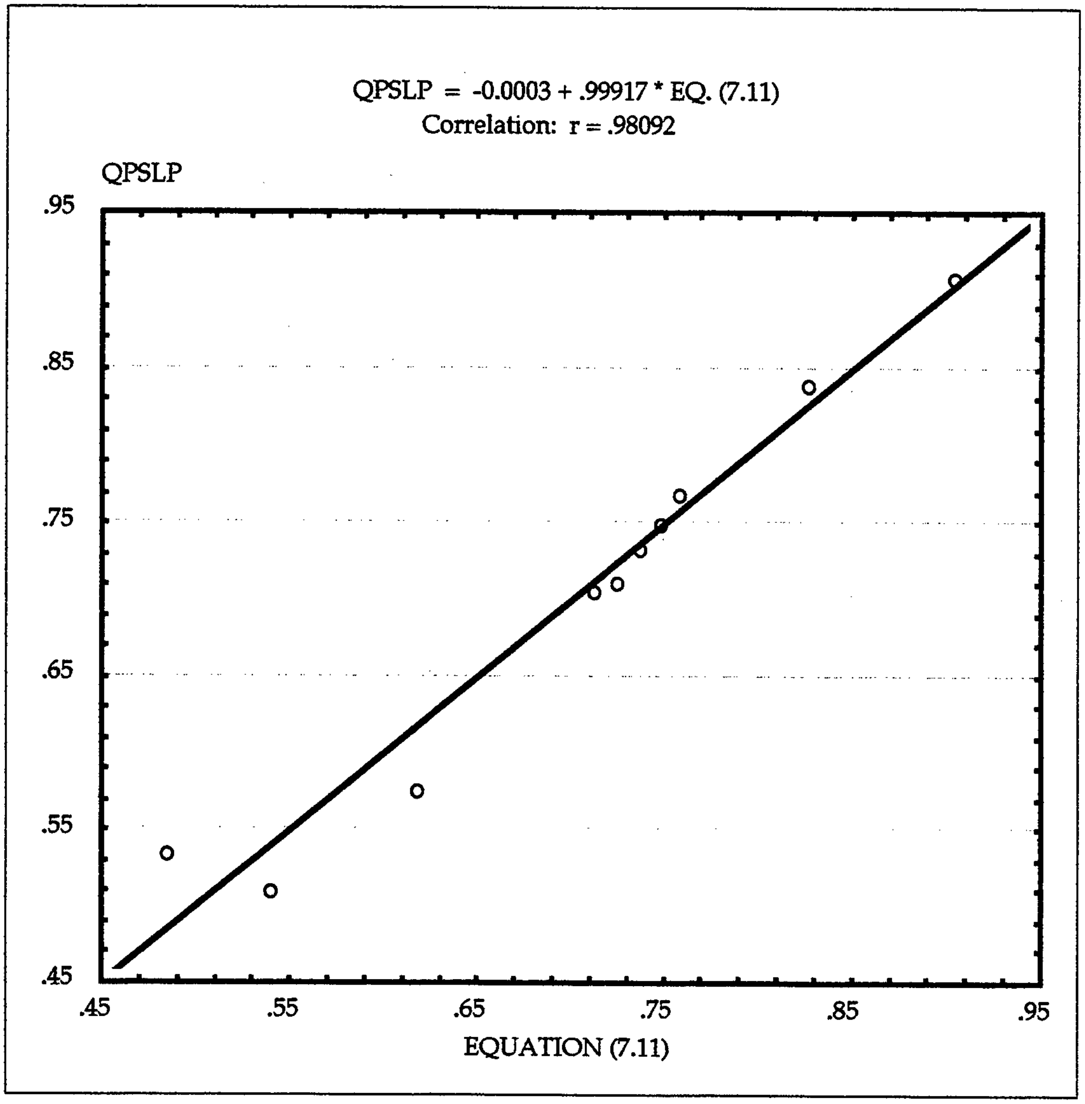

Figure 18. Regression Results from Fitting Equation (7.11) to QPSLP Data.

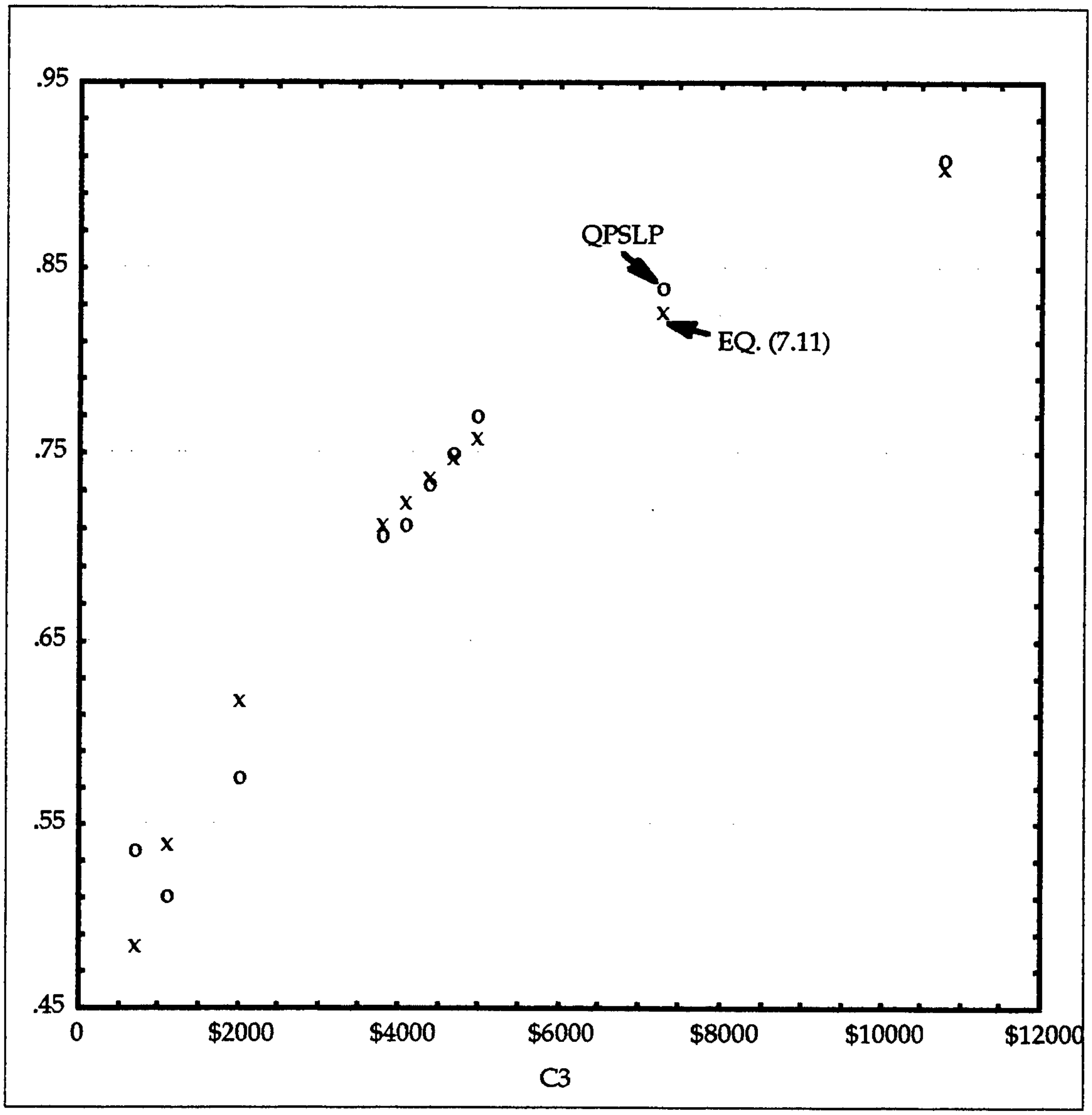

Figure 19. Comparison of Equation (7.11) to QPSLP Data for all C_3 Values.

For both Q_P and Q_R the effects of $PCLT$ and $RTAT$ were completely described by ZB so there are no formulas in the tables for these parameters. It is important to note that the formulas involving REP did include $REP = 0$, which corresponds to batch repair, in their development.

In the combining of the various effects the summing all of the constant terms gave values which was far from what should be the values to give a

reasonable slope and intercept for Q_P and Q_R for any set of values of parameters and a range of SW values from ZB to 150. Therefore, those constants were ignored and constants were determined by selecting several sets of data and comparing their known slopes and intercepts with those being provided by the sum of effects without constants included. An average value was then selected for the summed effects' constants. For Q_P the combined effects' slope constant was -2.392 and the intercept constant was -7.1.

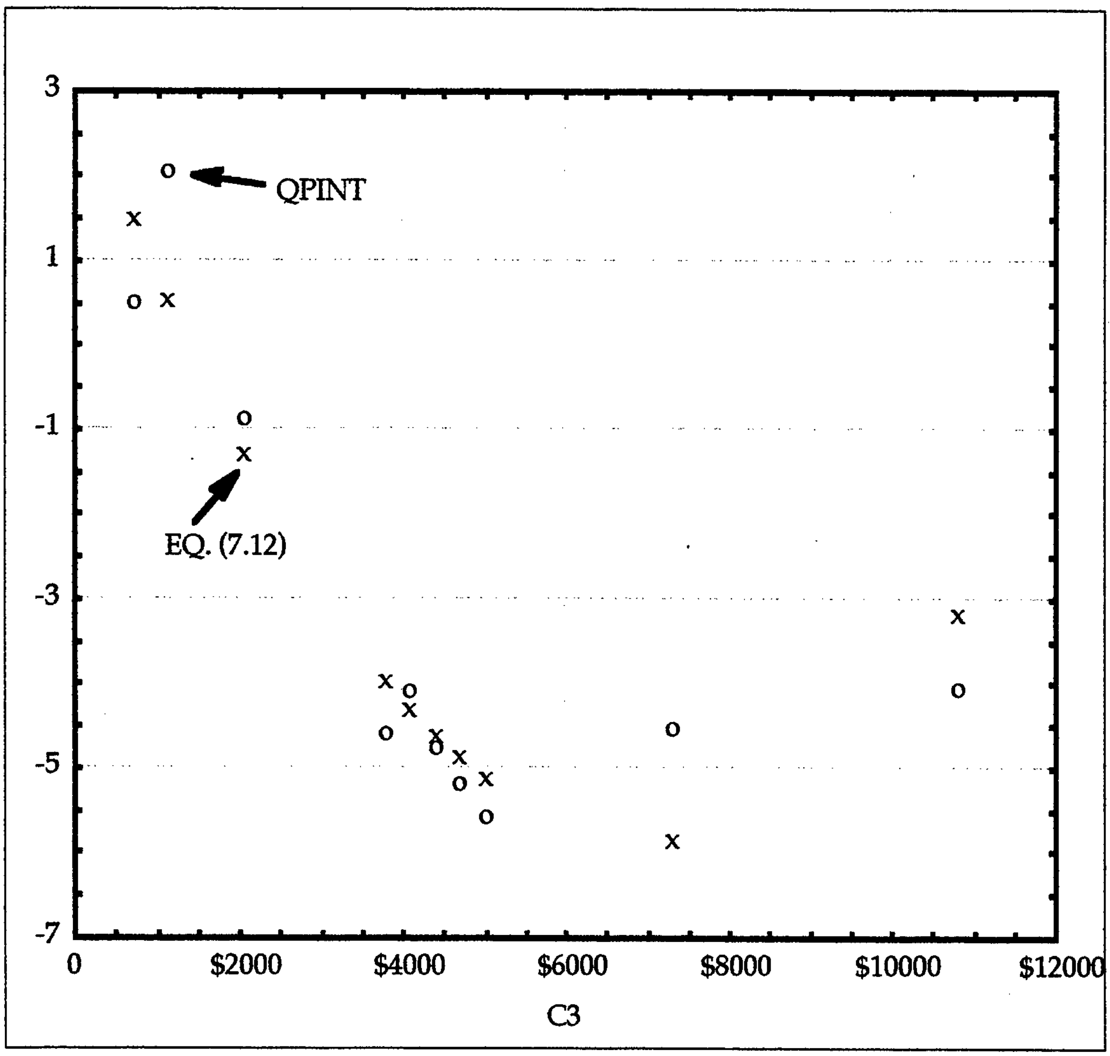

Figure 20. Comparison of Equation (7.12) to QPINT Data for all C_3 Values.

Table 1. Slope and Intercept Equations for Estimating Optimal Q_P.

Q_P SLOPE & INTERCEPT	Correlation
$QPSLP(D, CRR*RSR) = -1.04 + 1.67D^{0.01} + \left(-1.602 + \frac{1.57}{D^{0.1}}\right) CRR*RSR$	$r = 0.99268$
$QPINT(D, CRR*RSR) = -0.297 + D(0.19 - 0.008D) - D(1.76 - 0.08D)(CRR*RSR - 0.06D)^2$	$r = 0.78815$
$QPSLP(\lambda) = 0.01157 + \frac{2.403}{\lambda^{0.22}}$	$r = 0.98933$
$QPINT(\lambda) = 0.661 + 5.208 * \ln\left(\frac{\lambda}{1000}\right)$	$r = 0.95560$
$QPSLP(REP) = 0.6567 - 0.2927 REP^{0.35}$	$r = 0.99621$
$QPINT(REP) = -3.508 + 7.288 REP^{0.3}$	$r = 0.99020$
$QPSLP(A) = 0.469 + 0.00003A$	$r = 0.98735$
$QPINT(A) = 0.622 - 0.00026A$	$r = -0.5735$
$QPSLP(A_2) = 0.5257 + 0.00001A_2$	$r = 0.81992$
$QPINT(A_2) = 3.2027 - 0.00356A_2$	$r = -0.9805$

Table 2. Slope and Intercept Equations for Estimating Optimal Q_R.

Q_R *SLOPE & INTERCEPT*	Correlation
$QRSLP(C_3) = 0.31 + 0.45\left(\frac{C_3}{1000}\right)^{0.25}$	$r = 0.99152$
$QRINT(C_3) = -0.0656 - \left(\frac{1000}{55.1 + 0.0195C_3}\right)$	$r = 0.96689$
$QRSLP(D, CRR * RSR) = 0.4 + 0.36(CRR * RSR)^{0.55} + 0.045\sqrt{D}$	$r = 0.95497$
$QRINT(D, CRR * RSR) = -4.52 + D\left[-0.875 + 1.8(CRR * RSR - 0.35)^2\right]$	$r = 0.99776$
$QRSLP(\lambda) = 0.756 + \frac{31.65}{\lambda^{0.95}}$	$r = 0.99912$
$QRINT(\lambda) = -12.22 - 0.066\left(\frac{\lambda - 700}{100}\right)^2$	$r = 0.89309$
$QRSLP(REP) = 0.6337 + 0.3409\,REP^{0.35}$	$r = -0.9933$
$QRINT(REP) = -3.508 + 7.288\,REP^{0.3}$	$r = 0.99020$
$QRSLP(A) = 0.746 + 0.213e^{-\frac{A}{1000}}$	$r = 0.98816$
$QRINT(A) = -10.05 - 0.00196A$	$r = -0.9794$
$QRSLP(A_2) = 0.80614 - 0.00006A_2$	$r = -0.9898$
$QRINT(A_2) = -15.45 + 0.00253A_2$	$r = 0.99356$

The formula for estimating optimal Q_P when $SW \geq ZB$ is

$$QPEST = Q_P^*(SW=0) + \sum QPINT + \left[\sum QPSLP\right](SW - ZB), \tag{7.13}$$

where the Q_R value used in ZB when $REP > 0$ is the value of $QREST$ obtained from equation (7.15). The derivation of equation (7.15) begins with

$$\begin{aligned} QREST &= Q_R^*(SW=0) + \sum QRINT + \left[\sum QRSLP\right]\{SW - ZB(QREST)\} \\ &= Q_R^*(SW=0) + \sum QRINT + \left[\sum QRSLP\right]SW \\ &\quad - \left[\sum QRSLP\right]D(1 - CRR * RSR)PCLT \\ &\quad - \left[\sum QRSLP\right]D * CRR * RSR * RTAT \\ &\quad + \left[\sum QRSLP\right]CRR * RSR\frac{REP}{2} \\ &\quad - \left[\sum QRSLP\right]D * CRR * RSR\frac{REP}{2}QREST. \end{aligned} \tag{7.14}$$

Rewriting equation (7.14) by moving $QREST$ to the left-hand side of the equation,

$$\begin{aligned} & QREST + \left[\sum QRSLP\right]D * CRR * RSR\frac{REP}{2}QREST \\ &= Q_R^*(SW=0) + \sum QRINT + \left[\sum QRSLP\right]SW \\ &\quad - \left[\sum QRSLP\right]D(1 - CRR * RSR)PCLT \\ &\quad - \left[\sum QRSLP\right]D * CRR * RSR * RTAT \\ &\quad + \left[\sum QRSLP\right]CRR * RSR\frac{REP}{2} \\ &= Q_R^*(SW=0) + \sum QRINT + \left[\sum QRSLP\right]\left\{SW - PPV + CRR * RSR\frac{REP}{2}\right\}. \end{aligned}$$

Finally, the formula for $QREST$ is

$$QREST = \frac{Q_R^*(SW=0) + \sum QRINT + \left[\sum QRSLP\right]\left\{SW - PPV + CRR * RSR\frac{REP}{2}\right\}}{1 + \left[\sum QRSLP\right]D * CRR * RSR\frac{REP}{2}}. \tag{7.15}$$

The combined effects' constants for the summed $QRSLP$ and $QRINT$ effects were 0.142 and 6.23, respectively.

Figures 21 and 22 show the regression results when the computed and estimated values of optimal Q_P and Q_R for the 65 sets of data are compared.

C. Conclusions

The formulas developed in this chapter are much more complex that those currently by the UICP model. However, they do give the minimum average annual total variable costs to manage an inventory of a repairable item over a broad range of maximum inventory position values. They have a constant value when $SW \le ZB$ and increase approximately linearly with SW when $SW > ZB$. The UICP formulas (equation(7.1)) do not change with SW. Indeed, they are only approximately optimal for one value of the maximum inventory for any set of parameters. Figure 23 illustrates this fact with the curves of comparative TVC values for a set of parameter data.

These formulas were developed to eliminate the requirement to conduct a time-consuming search for optimal Q_P and Q_R for each value of SW when conducting a marginal analysis while trying to find a set of $SW_i, i = 1, n$ which will minimize the aggregate MSRT of a set of n repairable items.

Unfortunately, cost minimizing and MSRT minimizing are two different objectives. As one expects, increasing SW for a fixed set of Q_P and Q_R values

will reduce the MSRT for an item. However, changing Q_P and Q_R so that they minimize TVC as SW changes causes the MSRT to increase as SW increases beyond $SW = ZB$. Figures 24 and 25 show how TVC-optimal Q_P and Q_R change with SW and how MSRT changes with SW when these TVC-optimal Q_P and Q_R are used.

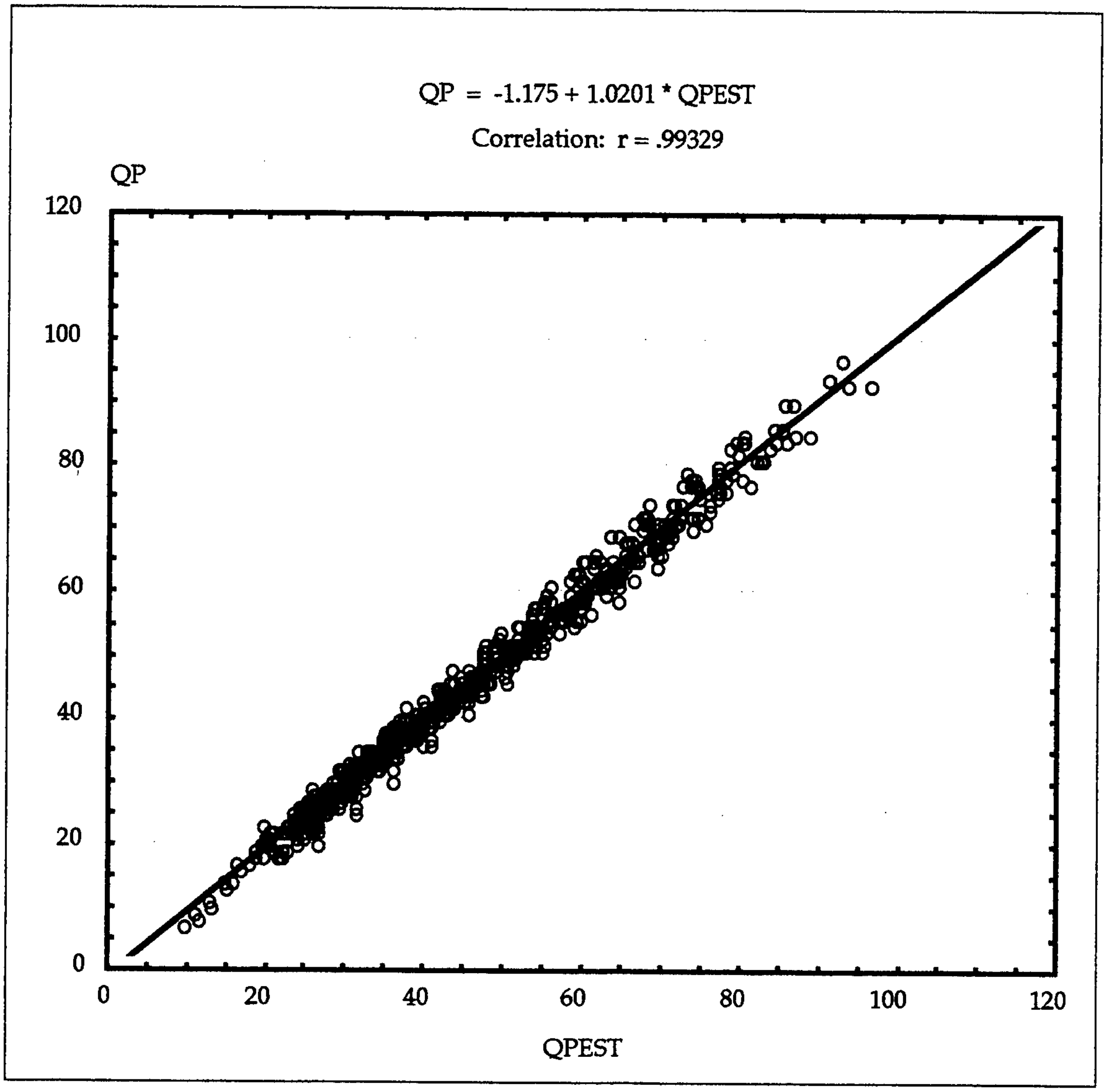

Figure 21. Regression Results from the Comparison of the Approximate Estimate of Optimal Q_P and the Computed Value of Optimal Q_P.

A companion report will show that attempting to use these TVC-optimal Q_P and Q_R values in the marginal analysis will result in a "less than optimal" aggregate MSRT because of the behavior shown in Figure 25. If, instead, Q_P and Q_R are fixed prior to the marginal analysis, then MSRT continues to decrease as SW increases and an optimal MSRT value can be obtained.

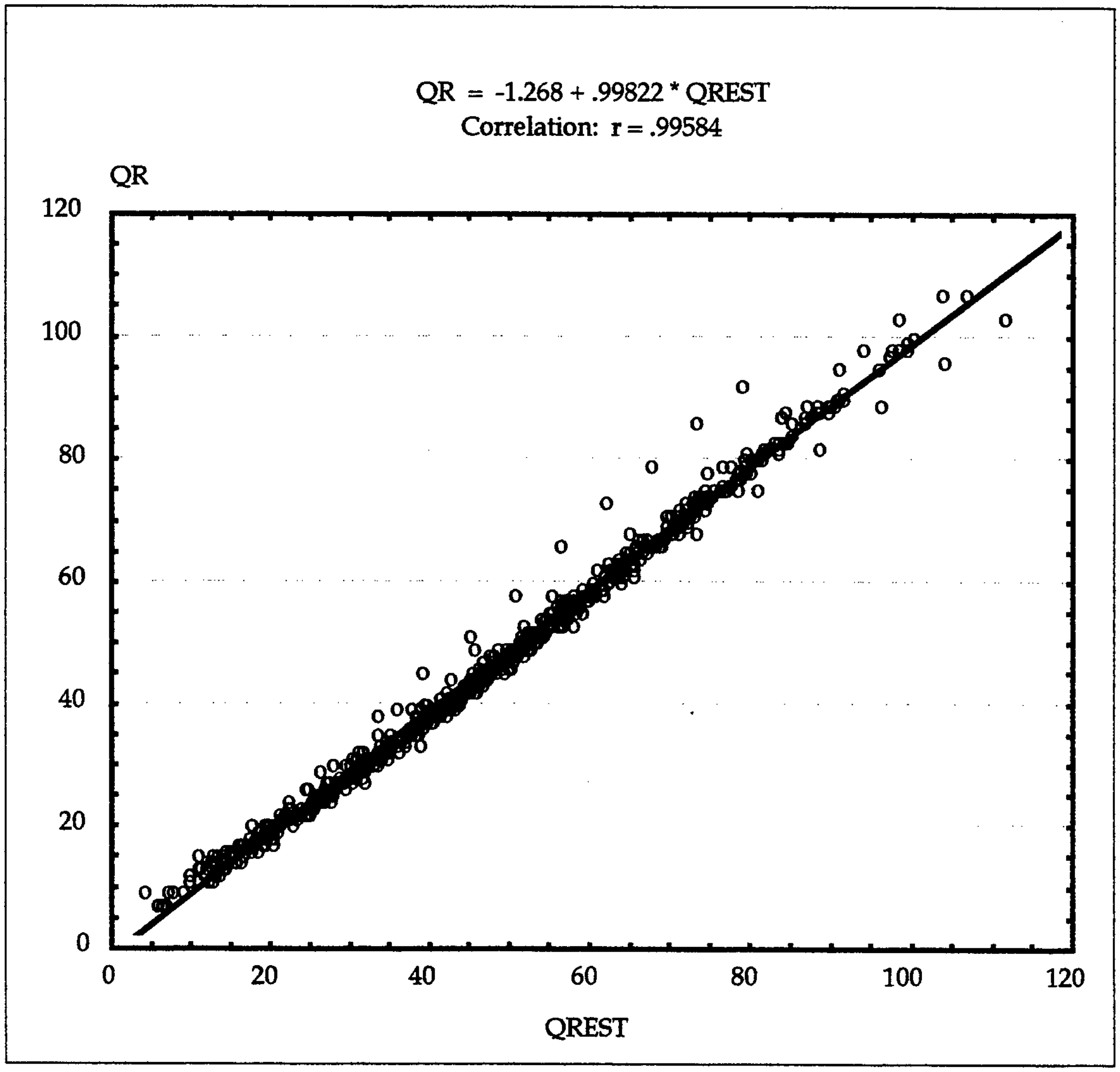

Figure 22. Regression Results from the Comparison of the Approximate Estimate of Optimal Q_R and the Computed Value of Optimal Q_R.

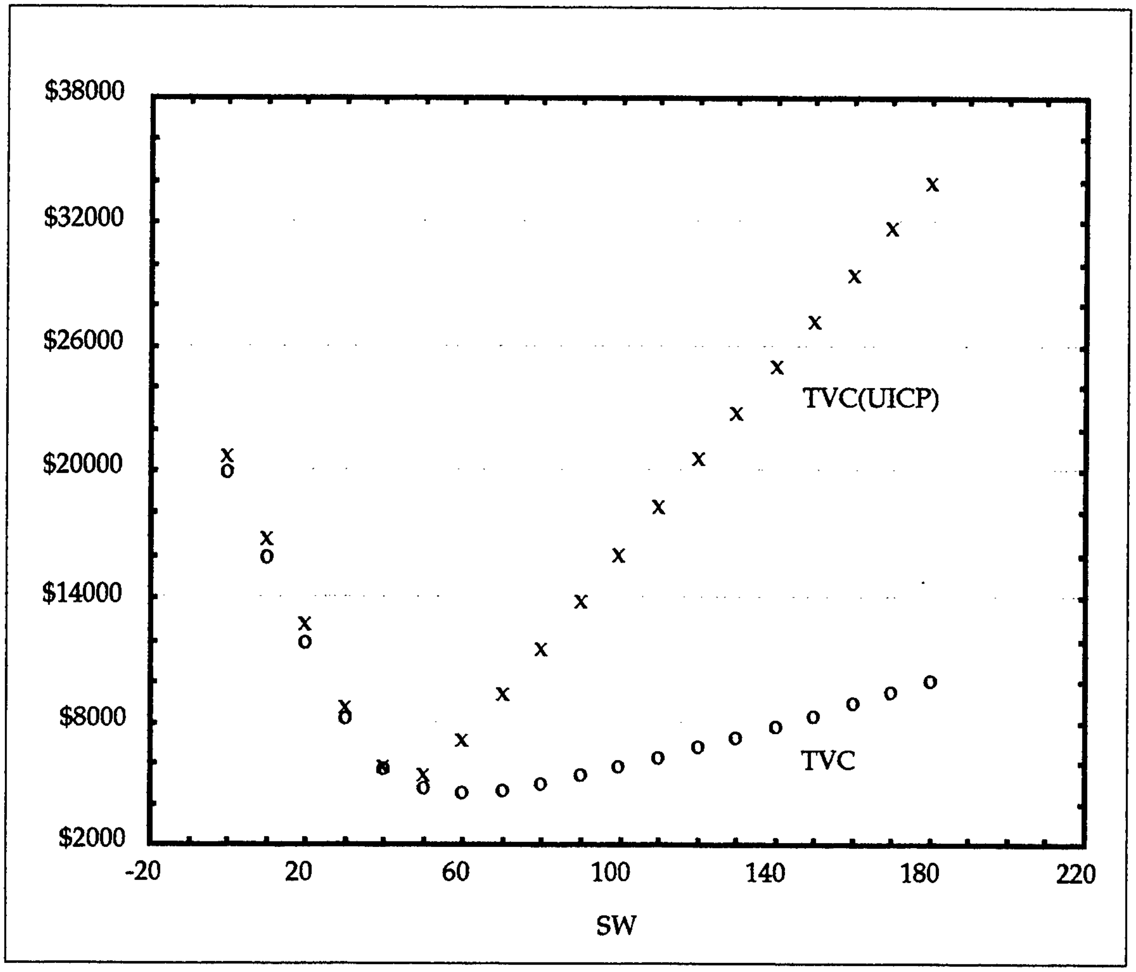

Figure 23. A Comparison of Optimal TVC and TVC having the UICP Values of Q_P and Q_R.

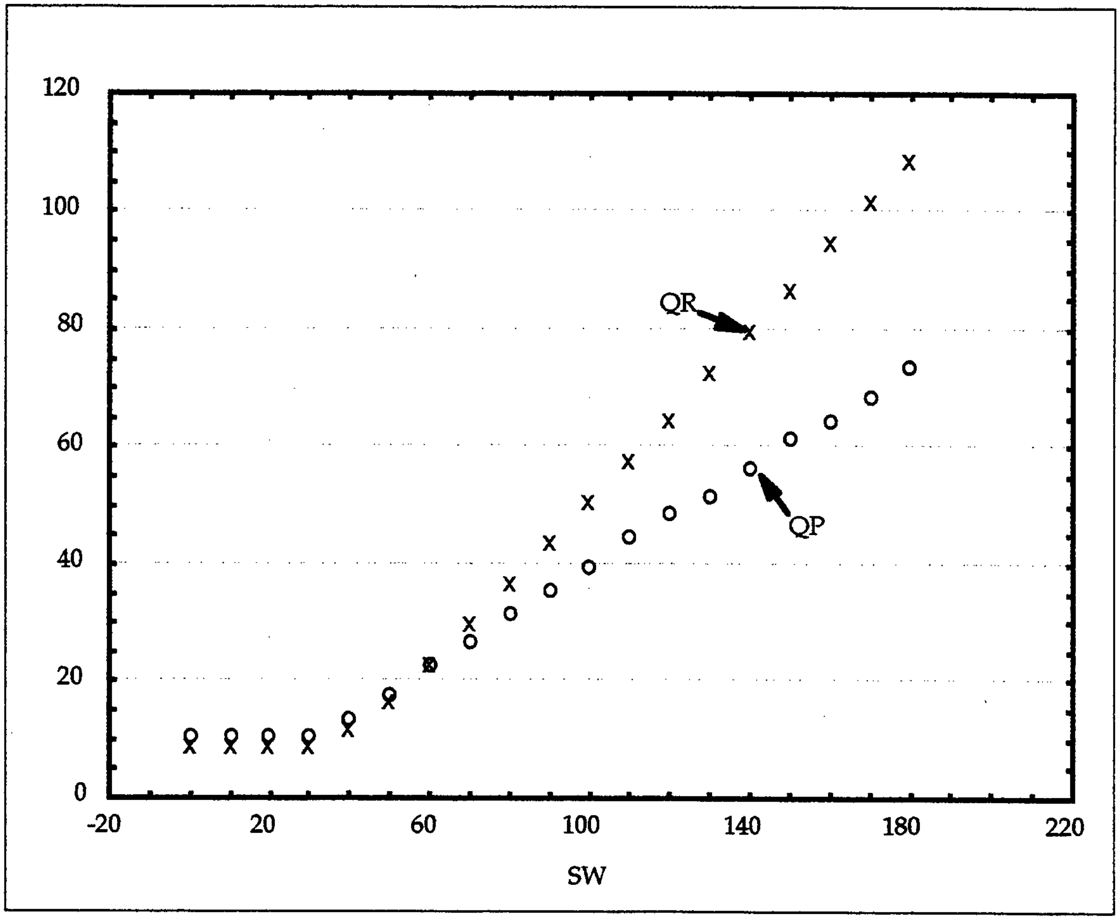

Figure 24. Min TVC -Optimal Q_P and Q_R for the Same Data as Figure 23.

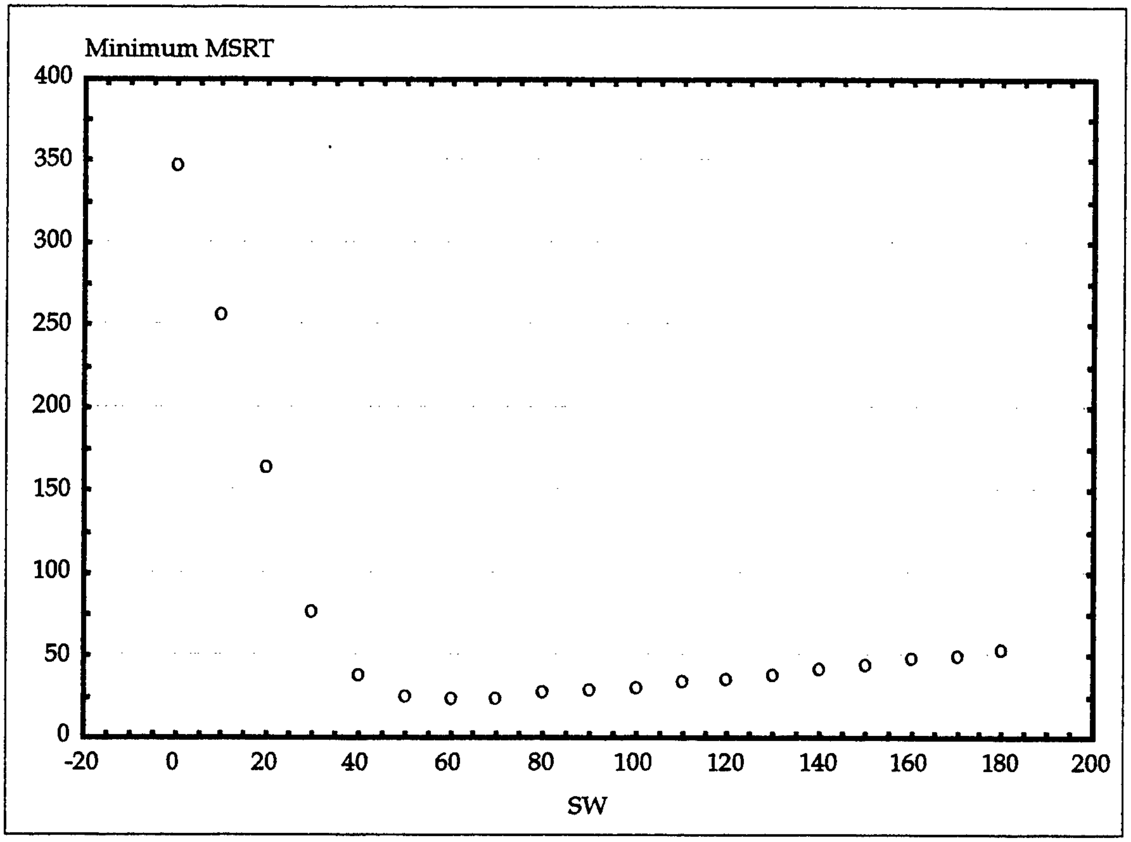

Figure 25. Minimum *MSRT* for Optimal Q_P and Q_R of Figure 23.

CHAPTER 8 - SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

A. Summary

Chapter 1 reviewed the history of the development of a new repairable inventory management model for the replenishment of stock, including the procurement of new units and repair of damaged carcasses. The goal of this effort is an analytical model of the Navy's repairable item inventory management process which will allow the determination of stockage depths which will meet readiness goals. This report presents the next logical steps in that development process.

Chapters 2 and 4 presented the derivations of formulas for the probability of being out of stock at any instant of time, $P_{OUT}(SW)$, for the cases of demand during aggregate lead time being Poisson and Normally distributed. These derivations provided formulas for all possible values of the maximum inventory position, SW. They assumed that values for the order and repair quantities, Q_P and Q_R, are given.

Chapters 3 and 5 presented the derivations of formulas for the expected number of backorders at any instant of time, $B(SW)$, for the cases of demand during aggregate lead time being Poisson and Normally distributed. As with the $P_{OUT}(SW)$ formulas, the derivations provided formulas for all possible values of SW and assumed that the values of Q_P and Q_R are given.

Chapter 6 presented the results of a simulation study to determine an approximate formula for the safety stock for repairable items for the case where demand during the aggregate lead time is Poisson distributed. Safety stock is

defined as the expected net inventory at the time of the arrival of a procurement order and/or the return of a repaired carcass from a depot. Four formulas were developed; two for the case of batched repair (no delays between inductions of units in the batch), and two for the case of a delay between units being inducted. These formulas are also dependent on knowing the values of SW, Q_P and Q_R.

Chapter 7 presents the approximate least cost formulas for Q_P and Q_R for the case where demand during the aggregate lead time is Poisson distributed. The formulas incorporate the effects of all model parameters. They are much more complex that the UICP formulas [5] and do increase linearly with SW when SW is greater than the expected demand during the aggregate lead time. The purpose of developing these formulas was to eliminate the time which would be required in determining their values during each step of a marginal analysis to find a set of optimal SW's for a group of items which minimize the aggregate MSRT value.

B. Conclusions

Formulas for $P_{OUT}(SW)$ and $B(SW)$ based on Baker's model [1] and an extension of it to the case of Normally distributed demand during aggregate lead time have been developed which should be easy to use in the Navy's Uniform Inventory Control Program (UICP) [5]. They are essential to the implementation of a readiness-based replenishment model. However, the formulas do depend on knowing the maximum inventory position SW, the order quantity Q_P, and the repair quantity Q_R.

Four approximate formulas for safety stock have been developed which are easy to use and intuitively appealing. However, they also require knowing

SW, Q_P, and Q_R, and the formulas of Chapter 6 are only valid for the Baker model.

The least cost formulas for the order and repair quantities are complex but do eliminate the need for an elaborate search routine to determine their values. for a given set of repairable item parameters. The major drawback to their use in a readiness-based model is that, because they increase with SW, they tend to cancel out the benefit (in the sense of reducing mean supply response time (MSRT)) which can be realized from using a fixed pair of Q_P and Q_R values.

C. Recommendations

The formulas for $P_{OUT}(SW)$ and $B(SW)$ are ready to be used by the UICP [5]. The problem which remains to be solved is how to determine the actual maximum inventory position (IP) for repairable item. A consumable item's maximum IP value can be determined for the sum of its reorder point and order quantity. This approach cannot be used for a repairable because it is impossible to determine a reorder point and it is not clear how the effects of Q_P and Q_R should be handled. Fortunately, an approximate value for SW can be found by using one of the formulas for safety stock.

Unfortunately, approximate formulas for safety stock for the case where the demand during the aggregate lead time is Normally distributed have not been derived yet. This case creates a problem for simulation modeling because the probability distribution for the time between demands is not known. One possible approach is to generate the quantity of demands for a quarter from the Normal distribution and then assume the arrival times are equally spaced over

the quarter. Hopefully, if enough quarters are simulated the result will be a good approximation to the assumed demand distribution.

The least cost formulas for Q_P and Q_R are not useful for a readiness-based replenishment model. As Figure 25 at the end of Chapter 7 shows, MSRT increases as the value of SW increases once SW exceeds the expected demand during the aggregate lead time, ZB. Fixing the Q_P and Q_R values at some value allows the value of MSRT to continue to decrease as SW increases. A study of the effects of several different fixed values of Q_P and Q_R is therefore needed. The results of a preliminary study of candidate Q_P and Q_R values are presented in a companion report.

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2. Boike, R. E., and Stringer, T. H., "An Evaluation of the Proposed MSRT Replenishment Model for Wholesale Consumable items," Masters Thesis, Naval Postgraduate School, December 1989.

3. Hadley, G., and Whitin, T. M., *Analysis of Inventory Systems*, Prentice-Hall, Englewood Cliffs, New Jersey, 1963.

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7. Richards, F. R., and McMasters, A. W., "Wholesale Provisioning Models: Model Development," Rpt No. NPS55-83-026, Naval Postgraduate School, September 1983.

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APPENDIX A - SIMAN IV [6] PROGRAM FOR SAFETY STOCK

A. MODEL FRAME

```
BEGIN, Repairable Model No.4;

;

next     CREATE: EX(1);

         COUNT: Demanded;

         ASSIGN: Inventory_position=Inventory_position-1;

         ASSIGN: Net_inventory=Net_inventory-1;

         BRANCH, 1: WITH, CRR, carcass:

            ELSE, no carcass;

;

no carcass ASSIGN: OrderQ=OrderQ + 1;

           BRANCH, 1:IF, OrderQ.GE.Q, order:

             ELSE, continue;

;

order  ASSIGN:OrderQ=0;

       TALLY:IP at order,Inventory_position;

        ASSIGN:Inventory_position=Inventory_position+Q;

        DELAY: PCLT;

        TALLY: Net inventory at receipt, Net_inventory;

        COUNT: Ordered,Q;

        ASSIGN: Net_inventory=Net_inventory+Q:DISPOSE;
```

```
;
carcass  QUEUE, Carcass;
         GROUP: QR;
         ASSIGN: Induct=0;
         ASSIGN: Repaired=0;
         TALLY: IP at order,Inventory_position;
         ASSIGN: Inventory_position=Inventory_position+QR;
         SPLIT;
         ASSIGN: Induct=Induct+1;
         BRANCH, 1: WITH, RSR, success:
             ELSE, bad carcass;
;
success ASSIGN: Repaired=Repaired+1;
         DELAY: (Repaired-1)*DELTA;
         BRANCH, 1:IF, DELTA.LE.0.0, repair2:
             ELSE, go on;
go on    DELAY: RTAT;
         TALLY: Net inventory at receipt, Net_inventory;
         COUNT: Repair;
         ASSIGN: Net_inventory=Net_inventory+1:DISPOSE;
;
repair2  BRANCH, 1:IF, Induct.EQ.QR, repair3:
            ELSE,continue;
;
```

```
repair3 ASSIGN: TempNR=Repaired;
        DELAY: RTAT;
        TALLY: Net inventory at receipt, Net_inventory;
        COUNT: Repair, TempNR;
        TALLY: Repair batch size, TempNR;
        ASSIGN: Net_inventory=Net_inventory+TempNR:DISPOSE;
;
bad carcass DELAY: Repaired*DELTA;
        ASSIGN: Inventory_position=Inventory_position-1;
        BRANCH, 1: IF,DELTA.EQ.0.0.AND.Induct.EQ.QR, check:
              ELSE, no carcass;
;
check   BRANCH, 1:IF, Repaired.GT.0,clone:
             ELSE, no carcass1;
;
no carcass1 ASSIGN: TempNR=0;
        TALLY: Repair batch size,TempNR:NEXT(no carcass);
;
clone   DUPLICATE: 1,repair3:NEXT(no carcass);
;
continue  DELAY:0: DISPOSE;
;
END;
```

B. EXPERIMENTAL FRAME

```
BEGIN;
PROJECT, Repairable Model No.4;
ATTRIBUTES: TempNR;
QUEUES: Carcass: Nocarcass;
PARAMETERS: 1,.05967;
VARIABLES: Demand, 16.76:
        PCLT, 6.07:
        RTAT, 1.28:
        DELTA, 0.:
        CRR, .9764:
        RSR, .85:
        Q, 6:
        QR, 16:
        OrderQ, 0:
        RepairQ, 0:
        Induct, 0:
        Repaired, 0:
        Num, 0:
        Inventory_position, 59:
        Net_inventory, 59;
TALLIES: Net inventory at receipt: Repair batch size: IP at order;
DSTATS: Net_inventory, Expected Net Inventory:Inventory_position, Exp IP;
COUNTERS: Demanded,,No:Ordered,,No:Repair,,No;
```

```
REPLICATE,,,1050,,,50;

END;
```

C. EXAMPLE OUTPUT LISTING

SIMAN IV - License #9050352

Naval Postgraduate School

Summary for Replication 1 of 1

Project: Repairable Model No.4 Run execution date : 8/ 9/1999

Replication ended at time : 1050.0

Statistics were cleared at time: 50.0

Statistics accumulated for time: 1000.0

TALLY VARIABLES

Identifier	Aver.	Var.	Min.	Max.	No. of Obs.
Net inventory at recei	8.7921	.76333	-22.00	33.000	1462
Repair batch size	13.670	.10017	8.0000	16.000	1007
IP at order	44.120	.13091	38.000	53.000	1461

DISCRETE-CHANGE VARIABLES

Identifier	Aver.	Var.	Min.	Max.	Final Value
Expected Net Inventory	14.876	.48251	-22.00	39.000	28.000
Exp IP	49.034	.10005	39.000	59.000	55.000

COUNTERS

Identifier	Count	Limit
Demanded	17297	Infinite
Ordered	2838	Infinite
Repair	14428	Infinite

Run Time: 1 hr(s) 53 min(s) and 30 sec(s)

Simulation run complete.

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6.	Professor Emeritus Alan W. McMasters, Code SM/Mg Department of Systems Management Naval Postgraduate School 555 Dyer Road, Rm. 209, Bldg. 330 Monterey, CA 93943-5103	3
7.	Commander, Naval Supply Systems Command Code SUP00 5450 Carlisle Pike Mechanicsburg, PA 17055	1
8.	Commander, Naval Inventory Control Point Code M0412 5450 Carlisle Pike P. O. Box 2020 Mechanicsburg, PA 17055-0788	3
9.	John Boyarski CACI 5010 Ritter Road, Suite 122 Mechanicsburg, PA, 17055	1